Unstable p-completion of motivic spaces

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§ Introduction

$X \in Sp \rightsquigarrow X^\wedge_p := \lim_n X/\!/p^n$ "p-adic completion"

$X/\!/p^n \simeq \mathrm{cofib}(X \xrightarrow{p^n} X)$

Can reconstruct X:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \prod_p X^\wedge_p \arrow[d] \\
X_{\mathbb{Q}} \arrow[r] & \left(\prod_p X^\wedge_p\right)_{\mathbb{Q}}
\end{tikzcd}$$

"arithmetic fracture square"

$(-)_{\mathbb{Q}}$ rationalization

$\rightsquigarrow$ Lots of tools to study $X^\wedge_p$, e.g. Adams SpecSeq

Similarly: $X \in An \rightsquigarrow X^\wedge_p \in An$

↳ Arithmetic fracture square

↳ Unstable Adams Spec Seq

Q: Is there an analog for other homotopy theories (e.g. in ∞-topoi, in motivic homotopy theory)?

§ Stable p-completion

$\mathcal{D}$ stable & presentable ∞-category

$\boxed{\approx \text{triangulated}}$ $\boxed{\exists \text{ all co/limits}}$

Def 1) $f: X \to Y$ in $\mathcal{D}$ is a **p-equivalence** $:\Leftrightarrow$ $f/\!\!/p : X/\!\!/p \longrightarrow Y/\!\!/p$ is an equivalence

2) $Z \in \mathcal{D}$ is **p-complete** $:\Leftrightarrow$ $\mathrm{Map}(Y, Z) \xrightarrow{f^*} \mathrm{Map}(X, Z)$ is an equivalence for all $f: X \longrightarrow Y$ p-equivalence

3) $\mathcal{D}_p^\wedge \subseteq \mathcal{D}$ full subcategory of p-complete objects

Lemma 1) The inclusion $\mathcal{D}_p^\wedge \subseteq \mathcal{D}$ has a left adjoint

$$(-)_p^\wedge : \mathcal{D} \longrightarrow \mathcal{D}_p^\wedge$$

"Bousfield Localization at the class of p-equivalences"

2) $(-)_p^\wedge \simeq \lim_n (-)/\!\!/p^n$

Lemma (Bousfield-Kan) For $\mathcal{D} = Sp$, $X \in Sp$, $n \in \mathbb{Z}$

there is a short exact sequence

$$0 \longrightarrow L_0\pi_n(X) \longrightarrow \pi_n(X_p^\wedge) \longrightarrow L_1\pi_{n-1}(X) \longrightarrow 0$$

$$L_i : Ab \xrightarrow{H} Sp \xrightarrow{(-)_p^\wedge} Sp \xrightarrow{\pi_n} Ab$$

"derived p-completion functors", $L_i = 0 \ \forall i \neq 0,1$

Cor $X \in Sp$, X n-connective $\Rightarrow$ $X_p^\wedge$ n-connective

Q Is this true for general $\mathcal{D}$?

Setting $\mathcal{D}$ equipped w/ a t-structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$,

write $\mathcal{D}^\heartsuit := \mathcal{D}_{\geq 0} \cap \mathcal{D}_{\leq 0}$ "standard heart of $\mathcal{D}$"

$$\pi_n : \mathcal{D} \longrightarrow \mathcal{D}^\heartsuit \qquad \text{"homotopy objects"}$$
$$X \longmapsto \Sigma^{-n}\tau_{\geq n}\tau_{\leq n}$$

A: No! E.g. if $X \in \mathcal{D}_{\geq 0}$ then $X_p^\wedge = \lim_n X/p^n \notin \mathcal{D}_{\geq 0}$

(there are examples of a $\mathcal{D}$, $X \in \mathcal{D}_{\geq 0}$ w/

$X_p^\wedge \notin \mathcal{D}_{\geq n}$ $\forall n \in \mathbb{Z}$)

Solution:

Def $\mathcal{D}^p_{\geq 0} := \{ X \in \mathcal{D} \mid X/\!/p \in \mathcal{D}_{\geq 0} \}$

$\mathcal{D}^p_{\leq -1} := \{ Z \in \mathcal{D} \mid \mathrm{Map}(X, Z) = 0 \ \forall X \in \mathcal{D}^p_{\geq 0} \}$

$(\mathcal{D}^p_{\geq 0}, \mathcal{D}^p_{\leq 0})$ is a t-structure, the

p-adic t-structure.

$\mathcal{D}^{p\heartsuit} := \mathcal{D}^p_{\geq 0} \cap \mathcal{D}^p_{\leq 0}$ p-adic heart

$\pi^p_n : \mathcal{D} \longrightarrow \mathcal{D}^{p\heartsuit}$, $X \longmapsto \Sigma^{-n} \tau^p_{\leq n} \tau^p_{\geq n} X$ p-adic homotopy objects

$\mathbb{L}_i : \mathcal{D}^\heartsuit \longrightarrow \mathcal{D}^{p\heartsuit}$, $X \longmapsto \pi^p_i (X^\wedge_p)$ derived p-completion

Compare to $L_i A = \pi_i A^\wedge_p$

Thm (M.) a) $\mathbb{L}_i = 0 \quad \forall i \neq 0, 1$ (not true for L_i)

b) For $X \in \mathcal{D}$, $n \in \mathbb{Z}$ there is a ses in $\mathcal{D}^{p\heartsuit}$

$$0 \longrightarrow \mathbb{L}_0 \pi_n(X) \longrightarrow \pi^p_n (X^\wedge_p) \longrightarrow \mathbb{L}_1 \pi_{n-1}(X) \quad 0$$

c) $X \in \mathcal{D}_{\geq 0} \Rightarrow X^\wedge_p \in \mathcal{D}^p_{\geq 0}$

"Bousfield-Kan holds in the p-adic t-structure"

<u>Rmk</u>: For $\mathcal{D} = Sp$, recover Bousfield–Kan:

$$Sp^{p\heartsuit} = Sp^{\heartsuit} \cap (Sp)_p^{\wedge} \subseteq Sp^{\heartsuit}$$

wrong in a general $\mathcal{D}$

$$L_i \simeq \mathbb{L}_i, \qquad \pi_n^p(X) \simeq \pi_n(X_p^{\wedge})$$

§ Unstable p-completion

$\mathcal{X}$ ∞-topos $\circ\circ\circ$ $\boxed{\mathcal{X} \simeq Shv_\tau(\mathcal{C})\ \text{sheaves of anima on a site}\ (\mathcal{C},\tau)}$

$\rightsquigarrow$ $Sp(\mathcal{X}) := \lim_n (\cdots \mathcal{X}_* \xrightarrow{\Omega} \mathcal{X}_* \xrightarrow{\Omega} \mathcal{X}_*)$

stable & presentable, "stabilization of $\mathcal{X}$"

$\Sigma^\infty_+ : \mathcal{X} \rightleftarrows Sp(\mathcal{X}) : \Omega^\infty$

__Def__ 1) $f: X \to Y$ in $\mathcal{X}$ is a p-equivalence $:\Leftrightarrow$ $\Sigma^\infty_+ f$ is a p-equivalence (i.e. $(\Sigma^\infty_+ f)/\!\!/p$ equivalence)

2) $Z \in \mathcal{X}$ is p-complete $:\Leftrightarrow$ $Map(Y,Z) \xrightarrow{f^*} Map(X,Z)$ is an equivalence for all $f: X \to Y$ p-equivalence

3) $\mathcal{X}_p^\wedge \subseteq \mathcal{X}$ full subcategory of p-complete objects

__Lemma__ $\mathcal{X}_p^\wedge \subseteq \mathcal{X}$ has a left adjoint $(-)_p^\wedge : \mathcal{X} \to \mathcal{X}_p^\wedge$

§ How to calculate p-completions

Problem: There is no easy formula "$X \mapsto \lim_n X/\!/p^n$"

So: How can we "calculate" $X_p^\wedge$?

Solution: Reduce to $Sp(\mathcal{X})$.

Lemma: Let $A \in Ab(\tau_{\leq 0}\mathcal{X})$, $n \geq 1$. Then

$$(K(A,n))_p^\wedge = (\Omega^\infty \Sigma^n HA)_p^\wedge \simeq \tau_{\geq 1}\Omega^\infty \Sigma^n (HA)_p^\wedge$$
$$= \tau_{\geq 1}\Omega^\infty \Sigma^n \left(\lim_k HA/\!/p^k\right)$$

Suppose $X \in \mathcal{X}_*$ is connected & n-truncated for some n

$\pi_0(X) = *,\quad \pi_k(X) = 0 \ \forall k > n$

There is a fiber sequence

$$K(\pi_n(X), n) \longrightarrow X \longrightarrow \tau_{\leq n-1} X$$

$\pi_1(X) = 0$

X simply-connected $\rightsquigarrow$ Can rotate

$$X \longrightarrow \tau_{\leq n-1} X \longrightarrow K(\pi_n(X), n+1)$$

A similar result holds more generally for nilpotent sheaves X. Everything from now on is also true for nilpotent X

Use this fiber sequence to get

Lemma $X_p^\wedge \simeq \tau_{\geq 1} \operatorname{fib}\Big((\tau_{\leq n-1} X)_p^\wedge \to \underbrace{K(\pi_n(X), n+1)_p^\wedge}_{\text{we know how to calculate this!}} \Big)$

Induction $\rightsquigarrow$ Can calculate $X_p^\wedge$ if X is n-truncated for some n.

Q What if X not truncated?

Def $\mathcal{X}$ is Postnikov-complete $:\Longleftrightarrow \mathcal{X} \xrightarrow{\sim} \lim_n \tau_{\leq n} \mathcal{X}$

In this case: $X \xrightarrow{\sim} \lim_n \tau_{\leq n} X$ for all $X \in \mathcal{X}$

Thm (M.) If $\mathcal{X}$ Postnikov-complete (+ some finiteness cond) then $X_p^\wedge \simeq \lim_n \underbrace{(\tau_{\leq n} X)_p^\wedge}_{\text{know how to calculate this!}}$ for all simply-connected $X \in \mathcal{X}_*$

Ex: Condition is satisfied in

An, $\mathcal{P}(\mathcal{C})$, $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k)$, $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$, ...

§ A short exact sequence for motivic spaces

Def k perfect field, Sm_k qc smooth k-schemes

1) $\{f_i : U_i \to U\}_{i\in I}$ is a Nisnevich cover $:\Leftrightarrow$
   - I finite
   - f_i is étale for $i \in I$
   - for $x \in U$ there is $i \in I$, $y \in U_i$, $f_i(y) = x$ and $k(y) \xrightarrow{\simeq} k(x)$ via f_i

Example:
- $\mathrm{Spec}(\mathbb{C}) \longrightarrow \mathrm{Spec}(\mathbb{C})$ Nis-cover
- $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ étale but not Nisnevich

2) $Shv_{nis}(Sm_k)$ ∞-topos of Nisnevich sheaves

Def 1) $X \in Shv_{nis}(Sm_k)$ is a motivic space (or $\mathbb{A}^1$-invariant) $:\Leftrightarrow$ $X(U) \xrightarrow{pr_U^*} X(U \times_{\mathrm{Spec}\,k} \mathbb{A}^1_k)$ is an equivalence for all $U \in Sm_k$

$\boxed{\mathbb{A}^1_k := \mathrm{Spec}(k[T])}$

2) $Spc(k) \subseteq Shv_{nis}(Sm_k)$ full subcategory of motivic spaces

Lemma: There is a left adjoint $L_{\mathbb{A}^1} : Shv_{nis}(Sm_k) \to Spc(k)$

WARNING $Spc(k)$ is not an ∞-topos & $L_{\mathbb{A}^1}$ is not left exact.

If $X \in Spc(k)$, then $X_p^\wedge$ means p-completion in $Shv_{nis}(Sm_k)$

Thm (M.) If $X \in Spc(k)_*$ is simply-connected, (in $Shv_{nis}(Sm_k)$)

then $X^\wedge_p \in Spc(k)_*$

"p-completion respects $\mathbb{A}^1$-invariance of simply-connected sheaves"

If $\mathcal{X} = An$, then Bousfield and Kan showed:

Thm (Bousfield–Kan) $X \in An_*$ simply-connected

For $n \geq 2$ there is a ses in $Sp^{p\heartsuit} \subseteq Sp^{\heartsuit}$

$$\begin{tikzcd} 0 \arrow[r] & L_0\pi_n(X) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \pi_n(X^\wedge_p) \arrow[r] & L_1\pi_{n-1}(X) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\ & \mathbb{L}_0\pi_n(X) & & \mathbb{L}_1\pi_{n-1}(X) & \end{tikzcd}$$

Goal Get a similar sequence for motivic spaces

Strategy: Look at

$$\begin{tikzcd} \mathcal{P}(W) \arrow[r, shift left, "L_\Sigma"] \arrow[r, phantom, "\perp"] & \mathcal{P}_\Sigma(W) \arrow[l, shift left] \arrow[r, phantom, "\simeq"] & Shv^{hyp}_{prozar}(ProZar(Sm_k)) \arrow[d, shift left, "\nu_*"] \arrow[d, phantom, "\dashv"] \\ Spc(k) \arrow[r, shift right] \arrow[r, phantom, "\perp"] & Shv_{nis}(Sm_k) \arrow[l, shift right, "L_{\mathbb{A}^1}"'] \arrow[r, shift right] \arrow[r, phantom, "\perp"] & Shv_{zar}(Sm_k) \arrow[l, shift right, "L_{nis}"'] \arrow[u, shift left, "\nu^*"] \end{tikzcd}$$

p-completion is "easy" here

p-completion is hard here

$$\begin{tikzcd}
\mathcal{P}(W) \arrow[r, shift left, "L_\Sigma"] \arrow[r, phantom, "\perp"] & \mathcal{P}_\Sigma(W) \arrow[l, hook, shift left] \arrow[r, phantom, "\cong"] & \mathrm{Shv}^{hyp}_{prozar}(\mathrm{ProZar}(\mathrm{Sm}_k)) \arrow[d, shift left, "\nu_*"] \arrow[d, phantom, "\dashv"] \\
\mathrm{Spc}(k) \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Shv}_{nis}(\mathrm{Sm}_k) \arrow[l, shift right, "L_{\mathbb{A}^1}"'] \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Shv}_{zar}(\mathrm{Sm}_k) \arrow[l, shift right, "L_{nis}"'] \arrow[u, hook, shift left, "\nu^*"]
\end{tikzcd}$$

Explain top row

schemes over Spec k

Def $\mathrm{ProZar}(\mathrm{Sm}_k)$ full subcategory of Sch_k consisting of schemes X such that there is a cofiltered diagram $X_\bullet : I \longrightarrow \mathrm{Sch}_k$ w/

a) $X_i \in \mathrm{Sm}_k$

b) $X = \lim\limits_{I} X_\bullet$

c) $X_i \longrightarrow X_j$ is a disjoint union of open immersions

Rmk $\mathrm{ProZar}(\mathrm{Sm}_k)$ is similar to $\mathrm{Pro}(\mathrm{Sm}_k)$, w/ the difference that $\mathrm{Spec}(\bar{k}) \notin \mathrm{ProZar}(\mathrm{Sm}_k)$ if $[\bar{k} : k] = \infty$

Def $\left.\begin{array}{l} f : X \longrightarrow Y \text{ in } \mathrm{ProZar}(\mathrm{Sm}_k) \\ \text{is pro-Zariski} \end{array}\right\} :\Longleftrightarrow \left\{\begin{array}{l} f = \lim\limits_{i \in I} f_i : \lim\limits_{i \in I} X_i \longrightarrow Y \\ \text{w/ } I \text{ cofiltered and} \\ X_i \to Y \text{ in } \mathrm{ProZar}(\mathrm{Sm}_k) \\ \text{a disjoint union of} \\ \text{open embeddings} \end{array}\right.$

Def $\mathrm{Shv}^{hyp}_{prozar}(\mathrm{ProZar}(\mathrm{Sm}_k))$ category of hypercomplete prozariski sheaves

covers are $\{f_i : U_i \to U\}_i$ st. $\{f_i\}_i$ is a fpqc-cover and f_i prozariski $\forall i$

ensures: $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$ iso $\forall n$ $\Rightarrow$ f equiv

"Whitehead's thm holds"

Thm There is a fully faithful geometric morphism

$$v^* : \mathrm{Shv}_{zar}(\mathrm{Sm}_k) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Shv}^{hyp}_{prozar}(\mathrm{ProZar}(\mathrm{Sm}_k)) : v_*$$

Thm: There is a subcategory $W \subseteq \mathrm{Shv}_{prozar}(\mathrm{ProZar}(\mathrm{Sm}_k))$ of "weakly contractible objects" such that

$$\mathrm{Shv}_{prozar}(\mathrm{ProZar}(\mathrm{Sm}_k)) \cong \mathcal{P}_\Sigma(W)$$

$$\mathcal{P}_\Sigma(W) \cong \mathrm{Shv}_{\amalg}(W) \simeq \left\{ F \in \mathrm{Fun}(W^{op}, \mathrm{An}) \,/\, F \text{ preserves finite products} \right\}$$

sheaves wrt the disjoint union topology, i.e. topology given by

$$\{U_i \to \coprod_{j \in I} U_j\}_{i \in I}$$

w/ I finite

Step 1: A short exact sequence for $\mathcal{P}(W)$

$$
\begin{tikzcd}
\mathcal{P}(W) \arrow[r, shift left, "L_\Sigma"] \arrow[r, phantom, "\perp"] & \mathcal{P}_\Sigma(W) \arrow[l, hook', shift left] \arrow[r, phantom, "\simeq"] & \mathrm{Shv}^{hyp}_{prozar}(\mathrm{ProZar}(\mathrm{Sm}_k)) \arrow[d, shift left, "\nu_*"] \arrow[d, phantom, "\dashv"] \\
\mathrm{Spc}(k) \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Shv}_{nis}(\mathrm{Sm}_k) \arrow[l, shift right, "L_{\mathbb{A}^1}"'] \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Shv}_{zar}(\mathrm{Sm}_k) \arrow[l, shift right, "L_{nis}"'] \arrow[u, shift left, "\nu^*"]
\end{tikzcd}
$$

$$\mathcal{P}(W) = \mathrm{Fun}(W^{op}, \mathrm{An})$$

Everything $((-)^\wedge_p, \pi_n(-), \mathbb{L}_i, p♡)$ is calculated levelwise

$\rightsquigarrow$ Levelwise Bousfield-Kan ses gives

$$0 \longrightarrow \mathbb{L}_0 \pi_n(X) \longrightarrow \pi_n(X^\wedge_p) \longrightarrow \mathbb{L}_1 \pi_{n-1}(X) \longrightarrow 0$$

ses in $\quad Sp(\mathcal{P}(W))^{p♡} \subseteq Sp(\mathcal{P}(W))^{♡}$

Step 2: A short exact sequence for $\mathcal{P}_\Sigma(W)$

$$
\begin{tikzcd}
\mathcal{P}(W) \arrow[r, shift left, "L_\Sigma"] \arrow[r, phantom, "\perp"] & \mathcal{P}_\Sigma(W) \arrow[l, hook', shift left] \arrow[r, phantom, "\simeq"] & \mathrm{Shv}^{hyp}_{prozar}(\mathrm{ProZar}(\mathrm{Sm}_k)) \arrow[d, shift left, "\nu_*"] \arrow[d, phantom, "\dashv"] \\
\mathrm{Spc}(k) \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Shv}_{nis}(\mathrm{Sm}_k) \arrow[l, shift right, "L_{\mathbb{A}^1}"'] \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Shv}_{zar}(\mathrm{Sm}_k) \arrow[l, shift right, "L_{nis}"'] \arrow[u, shift left, "\nu^*"]
\end{tikzcd}
$$

$$\iota_\Sigma : \mathcal{P}_\Sigma(W) \longrightarrow \mathcal{P}(W)$$

preserves p-equivalences and p-complete objects

$\rightsquigarrow$ $\quad (-)^\wedge_p, \mathbb{L}_i$ commute w/ ι_Σ

In particular: $\quad Sp(\mathcal{P}_\Sigma(W))^{p♡} \subseteq Sp(\mathcal{P}_\Sigma(W))^{♡}$,

and for $n \geq 2$, $X \in \mathcal{P}_\Sigma(W)_*$ simply-connected

get $\quad 0 \longrightarrow \mathbb{L}_0 \pi_n(X) \longrightarrow \pi_n(X^\wedge_p) \longrightarrow \mathbb{L}_1 \pi_{n-1}(X) \longrightarrow 0$

in $\quad Sp(\mathcal{P}_\Sigma(W))^{p♡}$

Step 3: A short exact sequence for $\mathrm{Shv}_{zar}(\mathrm{Sm}_k)$

$$\begin{tikzcd}
\mathcal{P}(W) \arrow[r, shift left, "L_\Sigma"] \arrow[r, phantom, "\perp"] & \mathcal{P}_\Sigma(W) \arrow[l, hook, shift left] \arrow[r, phantom, "\simeq"] & \mathrm{Shv}^{hyp}_{prozar}(\mathrm{ProZar}(\mathrm{Sm}_k)) \arrow[d, shift left, "\nu_*"] \\
\mathrm{Spc}(k) \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Shv}_{nis}(\mathrm{Sm}_k) \arrow[l, shift right, "L_{\mathbb{A}^1}"'] \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Shv}_{zar}(\mathrm{Sm}_k) \arrow[l, shift right, "L_{nis}"'] \arrow[u, shift left, "\nu^*"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

We are in the following situation:

$$\nu^* : \mathrm{Shv}_{zar}(\mathrm{Sm}_k) \rightleftarrows_{\perp} \mathcal{P}_\Sigma(W) : \nu_*$$

Problem: $Sp(\mathrm{Shv}_{zar}(\mathrm{Sm}_k))^{p\heartsuit} \not\subseteq Sp(\mathrm{Shv}_{zar}(\mathrm{Sm}_k))^{\heartsuit}$

Therefore cannot expect that there is

$$0 \longrightarrow \underbrace{\mathbb{L}_0\pi_n(X)}_{\in p\heartsuit} \longrightarrow \underbrace{\pi_n(X^\wedge_p)}_{\in \heartsuit} \longrightarrow \underbrace{\mathbb{L}_1\pi_{n-1}(X)}_{\in p\heartsuit} \longrightarrow 0$$

$\rightsquigarrow$ need a replacement for $\pi_n(X^\wedge_p)$

Def: $X \in \mathrm{Shv}_{zar}(\mathrm{Sm}_k)$

$$\pi_n^p(X) := \nu_*\big(\underbrace{\pi_n\big((\nu^*X)^\wedge_p\big)}_{\in Sp(\mathcal{P}_\Sigma(W))^{p\heartsuit}}\big) \in Sp(\mathrm{Shv}_{zar}(\mathrm{Sm}_k))$$

Lemma

a) $X \xrightarrow{f} Y$ p-equivalence $\Rightarrow$ $\pi_n^p(f)$ iso $\forall n \geq 2$

In particular: $\pi_n^p(X) \xrightarrow{\sim} \pi_n^p(X^\wedge_p)$

b) $X \xrightarrow{f} Y$ morphism of simply connected sheaves, $\pi_n^p(f)$ equivalence for all $n \geq 2$

$\Rightarrow$ f equivalence

"p-adic Whitehead's theorem"

Q: Is $\pi_n^p(X) \in Sp(Shv_{Zar}(Sm_k))^{p\heartsuit}$?

In general: no

But this is true if

$$\left(\mathbb{L}_1 \pi_n((\upsilon^* X)^\wedge_p)\right)/\!\!/p \in \text{ess im}(\upsilon^*)$$

Guarantees that if $A := \pi_n((\upsilon^* X)^\wedge_p)$ has pro-Zariski locally unbounded p-power torsion, then this "appears Zariski-locally"

In technical terms: If $U \in \text{ProZar}(Sm_k)$ w/ $x_n \in A(U)$ s.t. $x_0 = 0$, $p x_n = x_{n-1}$ then there is already $V \in Sm_k$, w/ $y_n \in A(V)$ s.t. $y_n|_U = x_n$, $p y_n = y_{n-1}$, $y_0 = 0$

Using this, we get a short exact sequence in $Sp(Shv_{Zar}(Sm_k))^{p\heartsuit}$

$$0 \to \mathbb{L}_0 \pi_n(X) \to \pi_n^p(X^\wedge_p) \to \mathbb{L}_1 \pi_{n-1}(X) \to 0$$

for all simply-connected X w/

$$\left(\mathbb{L}_1 \pi_n((\upsilon^* X)^\wedge_p)\right)/\!\!/p \in \text{ess im}(\upsilon^*)$$

<u>Q</u> Is this condition ever satisfied

<u>A</u>: Yes, by motivic spaces!

$$\begin{tikzcd} \mathcal{P}(W) \arrow[r, shift left, "L_\Sigma"] \arrow[r, phantom, "\perp"] & \mathcal{P}_\Sigma(W) \arrow[l, shift left] \arrow[r, phantom, "\simeq"] & \mathrm{Shv}^{hyp}_{prozar}(\mathrm{ProZar}(\mathrm{Sm}_k)) \arrow[d, shift left, "\nu_*"] \arrow[d, phantom, "\dashv"] \\ \mathrm{Spc}(k) \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Shv}_{nis}(\mathrm{Sm}_k) \arrow[l, shift right, "L_{\mathbb{A}^1}"'] \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Shv}_{zar}(\mathrm{Sm}_k) \arrow[l, shift right, "L_{nis}"'] \arrow[u, shift left, "\nu^*"] \end{tikzcd}$$

<u>Why?</u> $U \in \mathrm{Sm}_k$, U connected w/ generic point η

$X \in \mathrm{Spc}(k)$.

<u>Lemma</u> (Gabber) Let $k \geq 0$. Then

$$\underset{\in \mathrm{Sm}_k}{\left(\pi_n(X)/p^k\right)(U)} \hookrightarrow \underset{\in \mathrm{ProZar}(\mathrm{Sm}_k)}{\left(\pi_n(X)/p^k\right)(\eta)}$$ "Gersten injectivity"

Using this $(\tau\varepsilon)$

<u>Thm</u> (M.) Let $X \in \mathrm{Spc}(k)_*$ be simply-connected and $n \geq 2$.

There is a ses in $\mathrm{Sp}(\mathrm{Shv}_{nis}(\mathrm{Sm}_k))^{p\heartsuit}$

$$0 \to \mathbb{L}_0\pi_n(X) \to \pi_n^p(X_p^\wedge) \to \mathbb{L}_1\pi_{n-1}(X) \to 0$$