The derived ∞-category of Cartier Modules

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September 5, 2024

Abstract

For an endofunctor $F\colon \mathcal{C} \to \mathcal{C}$ on an (∞-)category $\mathcal{C}$ we define the ∞-category $\mathrm{Cart}(\mathcal{C}, F)$ of generalized Cartier modules as the lax equalizer of F and the identity. This generalizes the notion of Cartier modules on $\mathbb{F}_p$-schemes considered in [BB09]. We show that in favorable cases $\mathrm{Cart}(\mathcal{C}, F)$ is monadic over $\mathcal{C}$. If $\mathcal{A}$ is a Grothendieck abelian category and $F\colon \mathcal{A} \to \mathcal{A}$ is an exact and colimit-preserving endofunctor, we use this fact to construct an equivalence $\mathcal{D}(\mathrm{Cart}(\mathcal{A}, F)) \simeq \mathrm{Cart}(\mathcal{D}(\mathcal{A}), \mathcal{D}(F))$ of stable ∞-categories. We use this equivalence to construct a perverse t-structure on $\mathcal{D}^b_{\mathrm{coh}}(\mathrm{Cart}(\mathrm{QCoh}(X), F_*))$ for any Noetherian $\mathbb{F}_p$-scheme X with finite absolute Frobenius $F\colon X \to X$.

Contents

1 Introduction **2**

2 Definition of Cartier modules **6**

3 The induced t-structure on lax equalizers **11**

4 The left adjoint of the forgetful functor **16**

5 The derived ∞-category of Cartier modules **21**

6 Applications **26**

A Derived ∞-categories **29**

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1 Introduction

A distinguishing aspect of algebraic geometry over a field of positive characteristic $p > 0$ is the existence of the Frobenius endomorphism. A particular point of interest are modules with an action of the Frobenius.

If X is an $\mathbb{F}_p$-scheme then a (quasi-coherent) $\mathcal{O}_X$-module with a left action of the absolute Frobenius $F\colon X \to X$ is called a Frobenius module. They are related via a Riemann-Hilbert-type correspondence to p-torsion étale sheaves, cf. [EK18, BL17, BP09].

There is also a dual notion of Cartier modules (which are related to Frobenius modules via Grothendieck-Serre duality by a result of Baudin [Bau23, Theorem 4.2.7]) that are (quasi-coherent) $\mathcal{O}_X$-modules with a right action of the Frobenius, or equivalently pairs (M, κ_M) of an $\mathcal{O}_X$-module M and an $\mathcal{O}_X$-linear morphism $\kappa_M\colon F_*M \to M$. Their category $\mathrm{Cart}(X)$ was first considered by Anderson in [And00] and more thoroughly studied by Blickle and Böckle in [BB09, BB13].

In fact, their definition can be carried out in any (∞-)category $\mathcal{C}$ with an endofunctor $F\colon \mathcal{C} \to \mathcal{C}$.

Definition A (Definition 2.4)**.** We define the ∞-category $\mathrm{Cart}(\mathcal{C}, F)$ of *Cartier modules in $\mathcal{C}$ with respect to the functor F* as the pullback

$$\begin{array}{ccc} \mathrm{Cart}(\mathcal{C}, F) & \xrightarrow{U_{\mathcal{C}}} & \mathcal{C} \\ {\scriptstyle \kappa_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle (F, \mathrm{id}_{\mathcal{C}})} \\ \mathrm{Arr}(\mathcal{C}) & \xrightarrow[(\mathrm{ev}_0, \mathrm{ev}_1)]{} & \mathcal{C} \times \mathcal{C}, \end{array}$$

in Cat_∞, where $\mathrm{Arr}(\mathcal{C}) \coloneqq \mathrm{Fun}(\Delta^1, \mathcal{C})$ is the ∞-category of arrows in $\mathcal{C}$ and the functors $\mathrm{ev}_0, \mathrm{ev}_1\colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ are induced by the inclusions $\{0\} \hookrightarrow \Delta^1$ and $\{1\} \hookrightarrow \Delta^1$, respectively.

We recover the definition of [BB09] as $\mathrm{Cart}(X) \coloneqq \mathrm{Cart}(\mathrm{QCoh}(X), F_*)$ where F_* denotes the direct image functor along the absolute Frobenius F on X. In many parts of the theory, for example for defining cohomological operations for Cartier modules as in [BB13], it is necessary to look at the derived category of Cartier modules $h\mathcal{D}(\mathrm{Cart}(X))$. Here and in the following $\mathcal{D}(-)$ denotes the derived ∞-category and $h\mathcal{D}(-)$ denotes its homotopy category, i.e. the (ordinary) triangulated derived category. Note that there is a natural comparison functor

$$\Phi\colon h\mathcal{D}(\mathrm{Cart}(X)) \to \mathrm{Cart}(h\mathcal{D}(X), F_*)\,.$$

One would hope that this functor is an equivalence, but unfortunately it is not, as it fails to be faithful. This makes it hard to define functors into the (ordinary) derived category of Cartier modules, like for example the twisted inverse image functor $f^!$, which are not induced by corresponding functors on the level of abelian categories $\mathrm{Cart}(X) \to \mathrm{Cart}(Y)$.

The reason for the described caveat is that when constructing the category $\mathrm{Cart}(h\mathcal{D}(X), F_*)$ of Cartier modules in the derived category, one has to look at the category $\mathrm{Arr}(h\mathcal{D}(X))$ of 'homotopy commutative' arrows in $\mathcal{D}(X)$, which is not equivalent to $h\,\mathrm{Arr}(\mathcal{D}(X))$, the category of 'homotopy coherent' arrows in $\mathcal{D}(X)$, cf. [Lur17b, Section 1.2.6]. However, the situation is much better if we work directly in the realm of ∞-categories: Then it is true that there is an equivalence $\mathcal{D}(\mathrm{Arr}(\mathcal{A})) \simeq \mathrm{Arr}(\mathcal{D}(\mathcal{A}))$ for any Grothendieck abelian category $\mathcal{A}$, cf. for example the answer of Hoyois on mathoverflow [Hoy]. Hence, one expects that for the derived ∞-category one has in fact an equivalence

$$\mathcal{D}(\mathrm{Cart}(\mathcal{A}, F)) \simeq \mathrm{Cart}(\mathcal{D}(\mathcal{A}), \mathcal{D}(F))\,.$$

The goal of the present paper is to prove this.

Theorem B (Theorem 5.1)**.** *Let $\mathcal{A}$ be a Grothendieck abelian category and $F\colon \mathcal{A} \to \mathcal{A}$ an exact and colimit-preserving functor. In particular, F induces a functor $\mathcal{D}(F)\colon \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$. Then $\mathrm{Cart}(\mathcal{A}, F)$ is a Grothendieck abelian category and there is a canonical functor $\mathcal{D}(\mathrm{Cart}(\mathcal{A}, F)) \to \mathrm{Cart}(\mathcal{D}(\mathcal{A}), \mathcal{D}(F))$ which is an equivalence of ∞-categories. Moreover, both ∞-categories carry a natural t-structure and the equivalence is t-exact.*

If we apply this with $F = \mathrm{id}$, we recover the following well-known result about the derived ∞-category of the endomorphism category $\mathrm{End}(\mathcal{A})$ of a Grothendieck abelian category $\mathcal{A}$ (cf. [BGT16, Definition 3.5] for the definition of the endomorphism category of an ∞-category).

Corollary C (Corollary 5.7)**.** *There is a canonical equivalence of ∞-categories $\mathcal{D}(\mathrm{End}(\mathcal{A})) \to \mathrm{End}(\mathcal{D}(\mathcal{A}))$.*

On the other hand, if we apply Theorem B with $\mathcal{A} = \mathrm{QCoh}(X)$ and $F = F_*$ the (exact) Frobenius pushforward functor, then we obtain the result about classical Cartier modules that was the motivation for this paper.

Corollary D (Corollary 6.2)**.** *There is a canonical equivalence of ∞-categories $\mathcal{D}(\mathrm{Cart}(X)) \to \mathrm{Cart}(\mathcal{D}(X), F_*)$.*

Moreover, for $\mathcal{A} = \mathrm{QCoh}(X)$ and $F = F^*$ with X a regular Noetherian $\mathbb{F}_p$-scheme, we get a corresponding result about classical Frobenius modules.

Corollary E (Corollaries 2.10 and 6.3)**.** *There is a canonical equivalence of ∞-categories $\mathcal{D}(\mathrm{Frob}(X)) \to \mathrm{Cart}(\mathcal{D}(X), F^*)$.*

We will use the present results in upcoming work [MW] where we will construct coherent functor formalisms[1] of derived Cartier modules in appropriate geometric settings. This will in particular include a construction of the twisted inverse image functor $f^!$.

As a main ingredient for the proof of Theorem B we prove the following result which could be of independent interest.

[1] We chose this name since the authors are currently unaware how many of the six functors exist but are confident that Cartier modules do *not* admit a full six functor formalism.

Theorem F (Theorem 4.6)**.** *Let $\mathcal{C}$ be an ∞-category that admits countable coproducts and let $F\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor that preserves them. Then the forgetful functor $U\colon \operatorname{Cart}(\mathcal{C}, F) \to \mathcal{C}$ exhibits $\operatorname{Cart}(\mathcal{C}, F)$ as monadic over $\mathcal{C}$ (in the sense of [Lur17a, Definition 4.7.3.4]).*

Recall that a monadic functor $U\colon \mathcal{C} \to \mathcal{D}$ abstracts the setting of the forgetful functor $U\colon \operatorname{Mod}_R \to \operatorname{Set}$ for a ring R. Precisely, a monadic functor is a conservative functor with a left adjoint (in the case of modules this is given by the free module functor), such that U commutes with certain colimits. Monadic functors are related to monads (i.e. monoid objects in an (∞-)category of endofunctors with respect to the monoidal structure given by composition) in the following way: Every adjunction $L \dashv R$ gives rise to a monad RL, with multiplication given by $RL \circ RL = R \circ LR \circ L \xrightarrow{\text{counit}} RL$, and unit $\mathrm{id} \xrightarrow{\text{unit}} RL$. In particular, every monadic functor U gives rise to a monad UL. The Barr-Beck Theorem (cf. [Lur17a, Theorem 4.7.3.5] for its formulation for ∞-categories) provides a partial converse to this construction: every monad $T\colon \mathcal{C} \to \mathcal{C}$ has a unique (up to contractible choice) monadic functor $U\colon \mathcal{D} \to \mathcal{C}$ with left adjoint L such that $T \simeq UL$.

In the proof of Theorem B, we first show that both ∞-categories in question are monadic over the base ∞-category, and then prove that the corresponding monads agree, thus giving an equivalence of the "module"-categories.

Recall that there is a perverse t-structure on $\mathcal{D}^b_{\mathrm{coh}}(X)$ for X a Noetherian scheme that admits a dualizing complex, cf. [BBD82, Gab04, AB21]. Baudin constructed in [Bau23, Definition 5.2.1] a corresponding perverse t-structure on coherent Cartier modules. We use Theorem B to give a more conceptual construction.

Theorem G (Theorem 6.9)**.** *Let X be a Noetherian $\mathbb{F}_p$-scheme with finite absolute Frobenius that admits a dualizing complex. The perverse t-structure on $\mathcal{D}^b_{\mathrm{coh}}(X)$ induces a perverse t-structure on $\mathcal{D}^b_{\mathrm{coh}}(\operatorname{Cart}(X))$ such that the forgetful functor is t-exact for these t-structures.*

This t-structure coincides with the t-structure of [Bau23, Definition 5.2.1].

Outline

In Section 2 we define quite generally the ∞-category $\operatorname{LEq}(F, G)$ for functors $F, G\colon \mathcal{C} \to \mathcal{D}$; and its incarnations $\operatorname{Cart}(\mathcal{C}, F)$ and $\operatorname{Frob}(\mathcal{C}, F)$. Moreover, we show that properties of $\mathcal{C}$ and $\mathcal{D}$ like stability or presentability, together with appropriate conditions on F and G, imply the same properties for $\operatorname{LEq}(F, G)$ (resp. for $\operatorname{Cart}(\mathcal{C}, F)$ or $\operatorname{Frob}(\mathcal{C}, F)$) and the forgetful functor $U\colon \operatorname{LEq}(F, G) \to \mathcal{C}$, cf. Proposition 2.6. In particular, we prove the first part of Theorem B that states that $\operatorname{Cart}(\mathcal{A}, F)$ is a Grothendieck abelian category.

In Section 3, if $\mathcal{C}$ and $\mathcal{D}$ are stable ∞-categories with t-structures such that the functors $F, G\colon \mathcal{C} \to \mathcal{D}$ are exact and F is right t-exact and G is left t-exact, we introduce a t-structure on $\operatorname{LEq}(F, G)$ that is characterized by the property that $U\colon \operatorname{LEq}(F, G) \to \mathcal{C}$ is t-exact, cf. Proposition 3.3. Moreover, we identify the heart as $\operatorname{LEq}(F^\heartsuit, G^\heartsuit)$ in Proposition 3.4.

In Section 4 we explicitly construct the left adjoint of the forgetful functor $U\colon \mathrm{Cart}(\mathcal{C}, F) \to \mathcal{C}$ and derive t-exactness results about it, cf. Corollary 4.8. Moreover, we show that the existence of a left adjoint of U implies that U exhibits $\mathrm{Cart}(\mathcal{C}, F)$ as monadic over $\mathcal{C}$ if $\mathcal{C}$ is a presentable ∞-category and F preserves colimits, cf. Theorem 4.6. This is one of the key ingredients for the proof of Theorem B.

In Section 5 we state and prove our main theorem, cf. Theorem 5.1. Roughly, by a consequence of the ∞-categorical Barr-Beck Theorem [Lur17a, Theorem 4.7.3.5] it is enough to check the following:

(1) The functor $\mathcal{D}(U_{\mathcal{A}})\colon \mathcal{D}(\mathrm{Cart}(\mathcal{A}, F)) \to \mathcal{D}(\mathcal{A})$ induced by the forgetful functor $U_{\mathcal{A}}$ on $\mathrm{Cart}(\mathcal{A}, F)$ exhibits $\mathcal{D}(\mathrm{Cart}(\mathcal{A}, F))$ as monadic over $\mathcal{D}(\mathcal{A})$.

(2) The forgetful functor $U_{\mathcal{D}(\mathcal{A})}\colon \mathrm{Cart}(\mathcal{D}(\mathcal{A}), \mathcal{D}(F)) \to \mathcal{D}(\mathcal{A})$ exhibits the ∞-category $\mathrm{Cart}(\mathcal{D}(\mathcal{A}), \mathcal{D}(F))$ as monadic over $\mathcal{D}(\mathcal{A})$.

(3) Denote by $\mathcal{D}(L_{\mathcal{A}})$ and $L_{\mathcal{D}(\mathcal{A})}$ the left adjoints of the functors above. Then the natural map $U_{\mathcal{D}(\mathcal{A})}L_{\mathcal{D}(\mathcal{A})}(M) \to \mathcal{D}(U_{\mathcal{A}})\mathcal{D}(L_{\mathcal{A}})(M)$ is an equivalence for each object M of $\mathcal{D}(\mathcal{A})$.

The proof of (1) is an application of a more general result about induced functors on derived ∞-categories (Corollary A.8) once we know that $U_{\mathcal{A}}$ exhibits $\mathrm{Cart}(\mathcal{A}, F)$ as monadic over $\mathcal{A}$. But this, as well as (2), is already shown in Section 4. The key point in the proof of (3) is a reduction to the case where M is an object in the heart $\mathcal{D}(\mathcal{A})^{\heartsuit}$. To do so, we use the t-exactness properties of the forgetful functor and its left adjoint that we showed in the previous sections.

In Section 6 we specialize the discussion to the case where $\mathcal{A}$ is the category of quasi-coherent $\mathcal{O}_X$-modules over a Noetherian $\mathbb{F}_p$-scheme X. In particular, we construct a perverse t-structure on $\mathcal{D}^b_{\mathrm{coh}}(\mathrm{Cart}(\mathrm{QCoh}(X), F_*))$.

Furthermore, in Appendix A we collect some essentially well-known results about the (unbounded) derived ∞-category $\mathcal{D}(\mathcal{A})$ of a Grothendieck abelian category $\mathcal{A}$.

Notation

We freely use the language of ∞-categories as developed in [Lur17b, Lur17a]. In particular, we regard an (ordinary) category as an ∞-category via the nerve construction. Moreover, we denote by Cat_∞ the (very large) ∞-category of (large) ∞-categories.

Acknowledgements

We thank Manuel Blickle for suggesting the topic. We thank Tom Bachmann, Manuel Blickle, Anton Engelmann, Lorenzo Mantovani and Luca Passolunghi for helpful discussions and reading a draft of the article.

The authors acknowledge support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through the Collaborative Research Centre TRR 326 *Geometry and Arithmetic of Uniformized Structures*, project number 444845124.

2 Definition of Cartier modules

Let $\mathcal{C}$ be an ∞-category and $F\colon \mathcal{C} \to \mathcal{C}$ an endofunctor. In this section we define the ∞-category $\mathrm{Cart}(\mathcal{C}, F)$ of Cartier modules in $\mathcal{C}$ with respect to the functor F which is a generalization of the usual notion of Cartier modules. Moreover, we investigate how certain properties of $\mathcal{C}$ like stability or presentability induce the same properties on $\mathrm{Cart}(\mathcal{C}, F)$. We do the same for the ∞-category $\mathrm{Frob}(\mathcal{C}, F)$ of Frobenius modules in $\mathcal{C}$ with respect to F and for the more general concept of lax equalizers.

As $\mathrm{Cart}(\mathcal{C}, F)$ and $\mathrm{Frob}(\mathcal{C}, F)$ are special cases of lax equalizers recall the following definition.

Definition 2.1 ([NS18, Definition II.1.4])**.** Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors. The *lax equalizer of F and G* is the ∞-category $\mathrm{LEq}(F, G)$ defined as the pullback

$$\begin{array}{ccc} \mathrm{LEq}(F,G) & \xrightarrow{U} & \mathcal{C} \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle(F,G)} \\ \mathrm{Arr}(\mathcal{D}) & \xrightarrow[(\mathrm{ev}_0,\mathrm{ev}_1)]{} & \mathcal{D} \times \mathcal{D} \end{array}$$

in Cat_∞, where $\mathrm{Arr}(\mathcal{D}) \coloneqq \mathrm{Fun}(\Delta^1, \mathcal{D})$ is the ∞-category of arrows in $\mathcal{D}$ and the functors $\mathrm{ev}_0, \mathrm{ev}_1\colon \mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathcal{D}$ are induced by the inclusions $\{0\} \hookrightarrow \Delta^1$ and $\{1\} \hookrightarrow \Delta^1$, respectively.

Notation 2.2. Throughout the paper, in the situation above, we denote the natural maps by $U\colon \mathrm{LEq}(F, G) \to \mathcal{C}$ and $\kappa\colon \mathrm{LEq}(F, G) \to \mathrm{Arr}(\mathcal{D})$, respectively. We refer to the functor U as the *forgetful functor*.

Remark 2.3. *Using the universal property of the pullback we can describe the n-simplices of the ∞-category $\mathrm{LEq}(F, G)$:*

(a) Objects of $\mathrm{LEq}(F, G)$ are given by pairs (c, f) of an object c of $\mathcal{C}$ and an arrow $f\colon F(c) \to G(c)$ in $\mathcal{D}$.

(b) More generally, an n-simplex of $\mathrm{LEq}(F, G)$ is given by the data of an n-simplex $\alpha\colon \Delta^n \to \mathcal{C}$ of $\mathcal{C}$, a functor $\beta\colon \Delta^n \times \Delta^1 \to \mathcal{D}$ and equivalences $\beta|_{\Delta^n\times\{0\}} \simeq F\alpha$ and $\beta|_{\Delta^n\times\{1\}} \simeq G\alpha$.

Definition 2.4. Let $\mathcal{C}$ be an ∞-category and $F\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor. We define the ∞-category $\mathrm{Cart}(\mathcal{C}, F)$ of *Cartier modules in $\mathcal{C}$ with respect to the functor F* as $\mathrm{LEq}(F, \mathrm{id}_\mathcal{C})$ and the ∞-category $\mathrm{Frob}(\mathcal{C}, F)$ of *Frobenius modules in $\mathcal{C}$ with respect to the functor F* as $\mathrm{LEq}(\mathrm{id}_\mathcal{C}, F)$.

By Remark 2.3 we can describe the n-simplices of $\mathrm{Cart}(\mathcal{C}, F)$ and $\mathrm{Frob}(\mathcal{C}, F)$.

Remark 2.5. *Let $\mathcal{C}$ be an ∞-category and $F\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor.*

(a) Objects of the ∞-category $\mathrm{Cart}(\mathcal{C}, F)$ are given by pairs (M, κ_M) where $M \in \mathcal{C}$ is an object of $\mathcal{C}$ and $\kappa_M\colon F(M) \to M$ is a morphism in $\mathcal{C}$.

In particular, this generalizes the usual notion of Cartier modules on an $\mathbb{F}_p$-scheme X (cf. [BB09, Definition 2.1]) that are pairs (M, κ_M) of an $\mathcal{O}_X$-module M together with an $\mathcal{O}_X$-linear morphism $\kappa_M\colon F_*M \to M$ where $F\colon X \to X$ denotes the absolute Frobenius endomorphism of X.

(b) Similarly, objects of the ∞-category $\mathrm{Frob}(\mathcal{C}, F)$ are given by pairs (M, τ_M) where $M \in \mathcal{C}$ is an object of $\mathcal{C}$ and $\tau_M\colon M \to F(M)$ is a morphism in $\mathcal{C}$.

This generalizes the usual notion of Frobenius modules on an $\mathbb{F}_p$-scheme X (cf. [BL17, Remark 1.3.2]) that are pairs (M, τ_M) of an $\mathcal{O}_X$-module M together with an $\mathcal{O}_X$-linear morphism $\tau_M\colon M \to F_*M$ where $F\colon X \to X$ denotes the absolute Frobenius endomorphism of X.

(c) An n-simplex of $\mathrm{Cart}(\mathcal{C}, F)$ is given by a map $\beta\colon \Delta^n \times \Delta^1 \to \mathcal{C}$ and an equivalence $\beta|_{\Delta^n \times \{0\}} \simeq F\beta|_{\Delta^n \times \{1\}}$.

Similarly, an n-simplex of $\mathrm{Frob}(\mathcal{C}, F)$ is given by a map $\beta\colon \Delta^n \times \Delta^1 \to \mathcal{C}$ and an equivalence $\beta|_{\Delta^n \times \{1\}} \simeq F\beta|_{\Delta^n \times \{0\}}$.

The following proposition collects a number of useful facts about this construction.

Proposition 2.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors.*

(a) Let $M, N \in \mathrm{LEq}(F, G)$. The mapping space $\mathrm{Map}_{\mathrm{LEq}(F,G)}(M, N)$ is given by the equalizer

$$\mathrm{Eq}\left(\mathrm{Map}_{\mathcal{C}}(UM, UN) \underset{\kappa(N)_*F}{\overset{\kappa(M)^*G}{\rightrightarrows}} \mathrm{Map}_{\mathcal{D}}(FUM, GUN) \right).$$

(b) The forgetful functor $U\colon \mathrm{LEq}(F, G) \to \mathcal{C}$ is conservative, i.e. a morphism f in $\mathrm{LEq}(F, G)$ is an equivalence if and only if $U(f)$ is an equivalence in $\mathcal{C}$.

(c) Let $p\colon K \to \mathrm{LEq}(F, G)$ be a diagram such that the composite diagram $K \to \mathrm{LEq}(F, G) \to \mathcal{C}$ admits a limit and this limit is preserved by the functor G. Then p admits a limit and the functor $U\colon \mathrm{LEq}(F, G) \to \mathcal{C}$ preserves this limit.

(d) Let $p\colon K \to \mathrm{LEq}(F, G)$ be a diagram such that the composite diagram $K \to \mathrm{LEq}(F, G) \to \mathcal{C}$ admits a colimit and this colimit is preserved by the functor F. Then p admits a colimit and the functor $U\colon \mathrm{LEq}(F, G) \to \mathcal{C}$ preserves this colimit.

(e) *If $\mathcal{C}$ and $\mathcal{D}$ are stable ∞-categories (cf. [Lur17a, Definition 1.1.1.9]) and F and G are exact, then* $\mathrm{LEq}(F,G)$ *is a stable ∞-category and the functor* $U\colon \mathrm{LEq}(F,G)\to\mathcal{C}$ *is exact.*

(f) *If $\mathcal{C}$ is a presentable ∞-category (cf. [Lur17b, Definition 5.5.0.1]), $\mathcal{D}$ is an accessible ∞-category (cf. [Lur17b, Definition 5.4.2.1]), F preserves colimits and G is accessible (cf. [Lur17b, Definition 5.4.2.5]), then* $\mathrm{LEq}(F,G)$ *is a presentable ∞-category and the functor* $U\colon \mathrm{LEq}(F,G)\to\mathcal{C}$ *preserves colimits.*

(g) *If $\mathcal{C}$ and $\mathcal{D}$ are additive ∞-categories (cf. [GGN15, Definition 2.6]) and F and G are additive, then* $\mathrm{LEq}(F,G)$ *is an additive ∞-category and the functor* $U\colon \mathrm{LEq}(F,G)\to\mathcal{C}$ *is additive.*

(h) *If $\mathcal{C}$ and $\mathcal{D}$ are abelian categories and F and G are exact, then* $\mathrm{LEq}(F,G)$ *is an abelian category and the functor* $U\colon \mathrm{LEq}(F,G)\to\mathcal{C}$ *is exact.*

(i) *If $\mathcal{C}$ and $\mathcal{D}$ are Grothendieck abelian categories (i.e. they are presentable abelian and filtered colimits are exact) and F and G are exact and preserve colimits, then* $\mathrm{LEq}(F,G)$ *is a Grothendieck abelian category and the functor* $U\colon \mathrm{LEq}(F,G)\to\mathcal{C}$ *is exact and preserves colimits.*

Proof. Parts (a) to (f) are [NS18, Proposition II.1.5]. Note that (d) follows from their proof of (f).

For the proof of the remaining parts first note that all the claims about the functor U immediately follow from (c) and (d).

Using this there are only two things left to show in (g): that finite coproducts and finite products in $\mathrm{LEq}(F,G)$ are equivalent and that for each object M of $\mathrm{LEq}(F,G)$ the shear map

$$M\oplus M \xrightarrow{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)} M\oplus M$$

is an equivalence. But by (b) both of these can be checked after applying U where it is true because $\mathcal{C}$ is additive and U preserves finite coproducts and finite products (and in particular the shear map). This proves (g).

A similar argument, using finite colimits and finite limits instead of finite (co)products, and the comparison map between the coimage and the image, shows (h) (it is clear that a limit of 1-categories is still a 1-category, see e.g. [Lur17b, Proposition 2.3.4.12(4)]).

For (i) we have to show that $\mathrm{LEq}(F,G)$ is presentable and that filtered colimits in $\mathrm{LEq}(F,G)$ are exact. Presentability was already shown in (f). Checking that filtered colimits are exact can again be done after applying the functor U because it is conservative by (b), exact by (h) and colimit-preserving by (f). As filtered colimits in $\mathcal{C}$ are exact by assumption this completes the proof. □

Applying this with $\mathcal{D}=\mathcal{C}$ and F or G the identity we immediately get analogous results for Cartier modules and Frobenius modules.

Corollary 2.7. *Let $\mathcal{C}$ be an ∞-category and $F\colon\mathcal{C}\to\mathcal{C}$ be an endofunctor.*

(a) *Let* $M, N \in \mathrm{Cart}(\mathcal{C}, F)$. *The mapping space* $\mathrm{Map}_{\mathrm{Cart}(\mathcal{C},F)}(M, N)$ *is given by the equalizer*

$$\mathrm{Eq}\left(\mathrm{Map}_{\mathcal{C}}(UM, UN) \underset{\kappa(N)_* F}{\overset{\kappa(M)^*}{\rightrightarrows}} \mathrm{Map}_{\mathcal{C}}(FUM, UN) \right).$$

(b) *The forgetful functor* $U\colon \mathrm{Cart}(\mathcal{C}, F) \to \mathcal{C}$ *is conservative.*

(c) *Let* $p\colon K \to \mathrm{Cart}(\mathcal{C}, F)$ *be a diagram such that the composite diagram* $K \to \mathrm{Cart}(\mathcal{C}, F) \to \mathcal{C}$ *admits a limit. Then* p *admits a limit and the functor* $U\colon \mathrm{Cart}(\mathcal{C}, F) \to \mathcal{C}$ *preserves this limit.*

(d) *Let* $p\colon K \to \mathrm{Cart}(\mathcal{C}, F)$ *be a diagram such that the composite diagram* $K \to \mathrm{Cart}(\mathcal{C}, F) \to \mathcal{C}$ *admits a colimit and this colimit is preserved by the functor* F. *Then* p *admits a colimit and the functor* $U\colon \mathrm{Cart}(\mathcal{C}, F) \to \mathcal{C}$ *preserves this colimit.*

(e) *If* $\mathcal{C}$ *is a stable* ∞*-category and* F *is exact, then* $\mathrm{Cart}(\mathcal{C}, F)$ *is a stable* ∞*-category and the functor* $U\colon \mathrm{Cart}(\mathcal{C}, F) \to \mathcal{C}$ *is exact.*

(f) *If* $\mathcal{C}$ *is a presentable* ∞*-category and* F *preserves colimits, then* $\mathrm{Cart}(\mathcal{C}, F)$ *is a presentable* ∞*-category and the functor* $U\colon \mathrm{Cart}(\mathcal{C}, F) \to \mathcal{C}$ *preserves colimits.*

(g) *If* $\mathcal{C}$ *is an additive* ∞*-category and* F *is additive, then* $\mathrm{Cart}(\mathcal{C}, F)$ *is an additive* ∞*-category and the functor* $U\colon \mathrm{Cart}(\mathcal{C}, F) \to \mathcal{C}$ *is additive.*

(h) *If* $\mathcal{C}$ *is an abelian category and* F *is exact, then* $\mathrm{Cart}(\mathcal{C}, F)$ *is an abelian category and the functor* $U\colon \mathrm{Cart}(\mathcal{C}, F) \to \mathcal{C}$ *is exact.*

(i) *If* $\mathcal{C}$ *is a Grothendieck abelian category and* F *is exact and preserves colimits, then* $\mathrm{Cart}(\mathcal{C}, F)$ *is a Grothendieck abelian category and the functor* $U\colon \mathrm{Cart}(\mathcal{C}, F) \to \mathcal{C}$ *is exact and preserves colimits.*

Corollary 2.8. *Let* $\mathcal{C}$ *be an* ∞*-category and* $F\colon \mathcal{C} \to \mathcal{C}$ *be an endofunctor.*

(a) *Let* $M, N \in \mathrm{Frob}(\mathcal{C}, F)$. *The mapping space* $\mathrm{Map}_{\mathrm{Frob}(\mathcal{C},F)}(M, N)$ *is given by the equalizer*

$$\mathrm{Eq}\left(\mathrm{Map}_{\mathcal{C}}(UM, UN) \underset{\kappa(N)_*}{\overset{\kappa(M)^* F}{\rightrightarrows}} \mathrm{Map}_{\mathcal{C}}(FUM, UN) \right).$$

(b) *The forgetful functor* $U\colon \mathrm{Frob}(\mathcal{C}, F) \to \mathcal{C}$ *is conservative.*

(c) *Let* $p\colon K \to \mathrm{Frob}(\mathcal{C}, F)$ *be a diagram such that the composite diagram* $K \to \mathrm{Frob}(\mathcal{C}, F) \to \mathcal{C}$ *admits a limit and* F *preserves this limit. Then* p *admits a limit and the functor* $U\colon \mathrm{Frob}(\mathcal{C}, F) \to \mathcal{C}$ *preserves this limit.*

(d) *Let* $p\colon K \to \mathrm{Frob}(\mathcal{C}, F)$ *be a diagram such that the composite diagram* $K \to \mathrm{Frob}(\mathcal{C}, F) \to \mathcal{C}$ *admits a colimit. Then* p *admits a colimit and the functor* $U\colon \mathrm{Frob}(\mathcal{C}, F) \to \mathcal{C}$ *preserves this colimit.*

(e) *If* $\mathcal{C}$ *is a stable* ∞*-category and* F *is exact, then* $\mathrm{Frob}(\mathcal{C}, F)$ *is a stable* ∞*-category and the functor* $U\colon \mathrm{Frob}(\mathcal{C}, F) \to \mathcal{C}$ *is exact.*

(f) *If* $\mathcal{C}$ *is a presentable* ∞*-category and* F *is accessible, then* $\mathrm{Frob}(\mathcal{C}, F)$ *is a presentable* ∞*-category and the functor* $U\colon \mathrm{Frob}(\mathcal{C}, F) \to \mathcal{C}$ *preserves colimits.*

(g) *If* $\mathcal{C}$ *is an additive* ∞*-category and* F *is additive, then* $\mathrm{Frob}(\mathcal{C}, F)$ *is an additive* ∞*-category and the functor* $U\colon \mathrm{Frob}(\mathcal{C}, F) \to \mathcal{C}$ *is additive.*

(h) *If* $\mathcal{C}$ *is an abelian category and* F *is exact, then* $\mathrm{Frob}(\mathcal{C}, F)$ *is an abelian category and the functor* $U\colon \mathrm{Frob}(\mathcal{C}, F) \to \mathcal{C}$ *is exact.*

(i) *If* $\mathcal{C}$ *is a Grothendieck abelian category and* F *is exact and preserves colimits, then* $\mathrm{Frob}(\mathcal{C}, F)$ *is a Grothendieck abelian category and the functor* $U\colon \mathrm{Frob}(\mathcal{C}, F) \to \mathcal{C}$ *is exact and preserves colimits.*

We conclude this section by investigating the relation between Cartier modules and Frobenius modules with respect to adjoint functors.

Proposition 2.9. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be* ∞*-categories and* $F, G\colon \mathcal{C} \to \mathcal{D}$ *be functors. If* F *has a right adjoint* $R\colon \mathcal{D} \to \mathcal{C}$ *then there is an equivalence of* ∞*-categories*

$$\mathrm{LEq}(F, G) \simeq \mathrm{LEq}(\mathrm{id}_{\mathcal{C}}, RG) = \mathrm{Frob}(\mathcal{C}, RG)\,.$$

If G *has a left adjoint* $L\colon \mathcal{D} \to \mathcal{C}$ *then there is an equivalence of* ∞*-categories*

$$\mathrm{LEq}(F, G) \simeq \mathrm{LEq}(LF, \mathrm{id}_{\mathcal{C}}) = \mathrm{Cart}(\mathcal{C}, LF)\,.$$

Proof. Recall from [LT23, Section 2.1] that the oriented fiber product $\mathcal{C} \vec{\times}_{F,\mathcal{D},G} \mathcal{C}$ of F and G is defined as the pullback

$$\begin{array}{ccc} \mathcal{C} \vec{\times}_{F,\mathcal{D},G} \mathcal{C} & \longrightarrow & \mathcal{C} \times \mathcal{C} \\ \big\downarrow & & \big\downarrow{\scriptstyle (F,G)} \\ \mathrm{Arr}(\mathcal{D}) & \xrightarrow[(\mathrm{ev}_0, \mathrm{ev}_1)]{} & \mathcal{D} \times \mathcal{D} \end{array}$$

in Cat_∞. Hence, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{LEq}(F, G) & \xrightarrow{U} & \mathcal{C} \\ \big\downarrow & & \big\downarrow{\scriptstyle \Delta} \\ \mathcal{C} \vec{\times}_{F,\mathcal{D},G} \mathcal{C} & \longrightarrow & \mathcal{C} \times \mathcal{C} \\ \big\downarrow & & \big\downarrow{\scriptstyle (F,G)} \\ \mathrm{Arr}(\mathcal{D}) & \xrightarrow[(\mathrm{ev}_0, \mathrm{ev}_1)]{} & \mathcal{D} \times \mathcal{D} \end{array} \tag{1}$$

in Cat_∞, where Δ denotes the diagonal map. By definition of the lax equalizer $\mathrm{LEq}(F,G)$ and the oriented fiber product $\mathcal{C}\,\vec{\times}_{F,\mathcal{D},G}\,\mathcal{C}$ the outer rectangle and the lower square of (1) are cartesian diagrams. Thus, by the pasting law for pullbacks (cf. the dual of [Lur17b, Lemma 4.4.2.1]) the upper square of (1) is also a cartesian diagram.

If F has a right adjoint functor R, by [LT23, Lemma 2.2] there is a commutative diagram

$$\begin{array}{ccc} \mathcal{C}\,\vec{\times}_{F,\mathcal{D},G}\,\mathcal{C} & \xrightarrow{\quad\simeq\quad} & \mathcal{C}\,\vec{\times}_{\mathrm{id}_C,\mathcal{C},RG}\,\mathcal{C} \\ & \searrow \quad \swarrow & \\ & \mathcal{C}\times\mathcal{C} & \end{array}$$

where the horizontal arrow is an equivalence of ∞-categories. Combining this with the discussion above we get that

$$\begin{aligned} \mathrm{LEq}(F,G) &\simeq \lim\left(\mathcal{C} \xrightarrow{\Delta} \mathcal{C}\times\mathcal{C} \leftarrow \mathcal{C}\,\vec{\times}_{F,\mathcal{D},G}\,\mathcal{C}\right) \\ &\simeq \lim\left(\mathcal{C} \xrightarrow{\Delta} \mathcal{C}\times\mathcal{C} \leftarrow \mathcal{C}\,\vec{\times}_{\mathrm{id}_C,\mathcal{C},RG}\,\mathcal{C}\right) \\ &\simeq \mathrm{LEq}(\mathrm{id}_C, RG)\,. \end{aligned}$$

In the case where G has a left adjoint one argues analogously. □

Corollary 2.10. *Let $\mathcal{C}$ be an ∞-category and $L, R\colon \mathcal{C}\to\mathcal{C}$ be functors such that L is left adjoint to R. Then there is an equivalence of ∞-categories*

$$\mathrm{Cart}(\mathcal{C}, L) \simeq \mathrm{Frob}(\mathcal{C}, R).$$

3 The induced t-structure on lax equalizers

We fix the following notation for the whole section:

Let $\mathcal{C}$ be a stable ∞-category, cf. [Lur17a, Definition 1.1.1.9]. We denote the shift functor by $\Sigma\colon \mathcal{C}\to\mathcal{C}$. On objects $X\in\mathcal{C}$ it is given by $\Sigma(X) = 0 \amalg_X 0$, cf. [Lur17a, Notation 1.1.2.7].

Moreover, assume that $\mathcal{C}$ carries a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$, cf. [Lur17a, Definition 1.2.1.1]. This means that the following axioms are satisfied:

(a) $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ are subcategories of $\mathcal{C}$.

(b) For $X\in\mathcal{C}_{\geq 0}$ and $Y\in\mathcal{C}_{\leq 0}$ we have $\mathrm{Map}_{\mathcal{C}}(X, \Sigma^{-1}Y)\simeq 0$.

(c) $\Sigma\mathcal{C}_{\geq 0}\subseteq \mathcal{C}_{\geq 0}$ and $\Sigma^{-1}\mathcal{C}_{\leq 0}\subseteq\mathcal{C}_{\leq 0}$.

(d) For each $X\in\mathcal{C}$ there exists a fiber sequence $X'\to X\to X''$ with $X'\in\mathcal{C}_{\geq 0}$ and $X''\in\mathcal{C}_{\leq 0}$.

Note that we use homological notation for t-structures. As usual, for each n we denote by $\iota_{\geq n}\colon \mathcal{C}_{\geq n} \rightleftarrows \mathcal{C}\colon \tau_{\geq n}$ the adjunction of the inclusion and the connective cover functor and by $\tau_{\leq n}\colon \mathcal{C} \rightleftarrows \mathcal{C}_{\leq n}\colon \iota_{\leq n}$ the adjunction of the truncation and the inclusion functor, cf. [Lur17a, Proposition 1.2.1.5, Notation 1.2.1.7]. In particular, by [Lur17a, Remark 1.2.1.8] for every $X \in \mathcal{C}$ there is a fiber sequence

$$\iota_{\geq n}\tau_{\geq n}X \to X \to \iota_{\leq n-1}\tau_{\leq n-1}X\,.$$

Recall that the heart $\mathcal{C}^{\heartsuit}$ of the t-structure is given by $\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ and that there are homotopy object functors $\pi_n \coloneqq \tau_{\geq 0}\tau_{\leq 0}\Sigma^{-n}\colon \mathcal{C} \to \mathcal{C}^{\heartsuit}$, cf. [Lur17a, Definition 1.2.1.11].

Note that the ∞-category $\mathrm{Arr}(\mathcal{C})$ is also stable by [Lur17a, Proposition 1.1.3.1] and carries an induced t-structure given by $\mathrm{Arr}(\mathcal{C})_{\geq 0} = \mathrm{Arr}(\mathcal{C}_{\geq 0})$ and $\mathrm{Arr}(\mathcal{C})_{\leq 0} = \mathrm{Arr}(\mathcal{C}_{\leq 0})$. In particular, its heart is given by $\mathrm{Arr}(\mathcal{C})^{\heartsuit} = \mathrm{Arr}(\mathcal{C}^{\heartsuit})$.

Furthermore, let $\mathcal{D}$ be a stable ∞-category and $F, G\colon \mathcal{C} \to \mathcal{D}$ be exact functors. If $\mathcal{D}$ carries a t-structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ recall that F is called right t-exact (resp. left t-exact) if $F(\mathcal{C}_{\geq 0}) \subseteq \mathcal{D}_{\geq 0}$ (resp. $F(\mathcal{C}_{\leq 0}) \subseteq \mathcal{D}_{\leq 0}$).

In this section we introduce an induced t-structure on $\mathrm{LEq}(F, G)$. Note that this ∞-category is stable by Proposition 2.6(e). It is defined in such a way that the forgetful functor is automatically t-exact.

Definition 3.1. Define full subcategories $\mathrm{LEq}(F,G)_{\geq 0}$ and $\mathrm{LEq}(F,G)_{\leq 0}$ of $\mathrm{LEq}(F,G)$ as follows: An object $M \in \mathrm{LEq}(F,G)$ is in $\mathrm{LEq}(F,G)_{\geq 0}$ (resp. $\mathrm{LEq}(F,G)_{\leq 0}$) if and only if UM is in $\mathcal{C}_{\geq 0}$ (resp. in $\mathcal{C}_{\leq 0}$).

We prove in Proposition 3.3 that this defines a t-structure on $\mathrm{LEq}(F,G)$ if F is right t-exact and G is left t-exact. For the proof we need the following.

Lemma 3.2. *Let $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ be a t-structure on $\mathcal{D}$ such that $F\colon \mathcal{C} \to \mathcal{D}$ is right t-exact. Then there is a commutative diagram*

$$\begin{array}{ccc} F\iota_{\geq 0}\tau_{\geq 0} & \xrightarrow{Fc} & F \\ \downarrow & \nearrow_{cF} & \\ \iota_{\geq 0}\tau_{\geq 0}F & & \end{array}$$

where $c\colon \iota_{\geq 0}\tau_{\geq 0} \to \mathrm{id}$ denotes the counit of the adjunction.

Dually, if $G\colon \mathcal{C} \to \mathcal{D}$ is left t-exact then there is a commutative diagram

$$\begin{array}{ccc} \iota_{\geq 0}\tau_{\geq 0}G & \xrightarrow{cG} & G \\ \downarrow & \nearrow_{Gc} & \\ G\iota_{\geq 0}\tau_{\geq 0}\,. & & \end{array}$$

Proof. We only show the first claim. The second claim can be shown by a dual argument. There is a fiber sequence

$$\iota_{\geq 0}\tau_{\geq 0}F\iota_{\geq 0}\tau_{\geq 0} \xrightarrow{c} F\iota_{\geq 0}\tau_{\geq 0} \xrightarrow{u} \iota_{\leq -1}\tau_{\leq -1}F\iota_{\geq 0}\tau_{\geq 0}$$

where the last term is 0 by the right t-exactness of F. Hence, the counit induces an equivalence $\iota_{\geq 0}\tau_{\geq 0}F\iota_{\geq 0}\tau_{\geq 0} \simeq F\iota_{\geq 0}\tau_{\geq 0}$.

Moreover, the counit also yields a map $\iota_{\geq 0}\tau_{\geq 0}F\iota_{\geq 0}\tau_{\geq 0} \xrightarrow{\iota_{\geq 0}\tau_{\geq 0}Fc} \iota_{\geq 0}\tau_{\geq 0}F$. Putting everything together we obtain a diagram

$$\begin{array}{ccc} F\iota_{\geq 0}\tau_{\geq 0} & \xrightarrow{Fc} & F \\ {\scriptstyle cF\iota_{\geq 0}\tau_{\geq 0}}\uparrow{\scriptstyle \simeq} & & \\ \iota_{\geq 0}\tau_{\geq 0}F\iota_{\geq 0}\tau_{\geq 0} & & \uparrow{\scriptstyle cF} \\ {\scriptstyle \iota_{\geq 0}\tau_{\geq 0}Fc}\downarrow & & \\ \iota_{\geq 0}\tau_{\geq 0}F & \longrightarrow & \end{array}$$

that commutes because of the naturality of c. □

Proposition 3.3. *Assume that there is a t-structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ on $\mathcal{D}$ such that $F\colon \mathcal{C} \to \mathcal{D}$ is right t-exact and $G\colon \mathcal{C} \to \mathcal{D}$ is left t-exact. Then the full subcategories $\mathrm{LEq}(F,G)_{\geq 0}$ and $\mathrm{LEq}(F,G)_{\leq 0}$ define a t-structure on $\mathrm{LEq}(F,G)$.*

Moreover, the forgetful functor $U\colon \mathrm{LEq}(F,G) \to \mathcal{C}$ is t-exact.

Proof. We check that $(\mathrm{LEq}(F,G)_{\geq 0}, \mathrm{LEq}(F,G)_{\leq 0})$ satisfies the properties of a t-structure. First, we have to see that $\Sigma\,\mathrm{LEq}(F,G)_{\geq 0} \subseteq \mathrm{LEq}(F,G)_{\geq 0}$ and $\Sigma^{-1}\,\mathrm{LEq}(F,G)_{\leq 0} \subseteq \mathrm{LEq}(F,G)_{\leq 0}$. Let $M \in \mathrm{LEq}(F,G)_{\geq 0}$, i.e. $UM \in \mathcal{C}_{\geq 0}$. By Proposition 2.6(e) the functor U is exact, so we have $U\Sigma M \simeq \Sigma UM \in \mathcal{C}_{\geq 0}$ and hence $\Sigma M \in \mathrm{LEq}(F,G)_{\geq 0}$. The other inclusion can be shown analogously.

Next, let $M \in \mathrm{LEq}(F,G)_{\geq 0}$ and $N \in \mathrm{LEq}(F,G)_{\leq -1}$. We have to show that $\mathrm{Map}_{\mathrm{LEq}(F,G)}(M,N) = 0$. By Proposition 2.6(a) this mapping space can be computed as the equalizer of two maps $\mathrm{Map}_{\mathcal{C}}(UM,UN) \rightrightarrows \mathrm{Map}_{\mathcal{D}}(FUM,GUN)$. But we have $UM \in \mathcal{C}_{\geq 0}$ and $UN \in \mathcal{C}_{\leq -1}$ by definition and $FUM \in \mathcal{D}_{\geq 0}$ by the right t-exactness of F and $GUN \in \mathcal{D}_{\leq -1}$ by the left t-exactness of G. Thus, it follows that $\mathrm{Map}_{\mathcal{C}}(UM,UN) = 0$ and $\mathrm{Map}_{\mathcal{D}}(FUM,GUN) = 0$, hence $\mathrm{Map}_{\mathrm{LEq}(F,G)}(M,N) = 0$.

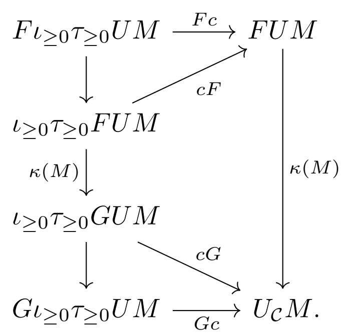
It remains to show that for any object $M \in \mathrm{LEq}(F,G)$ there is a fiber sequence $M' \to M \to M''$ with $M' \in \mathrm{LEq}(F,G)_{\geq 0}$ and $M'' \in \mathrm{LEq}(F,G)_{\leq -1}$. We have a fiber sequence

$$\iota_{\geq 0}\tau_{\geq 0}UM \xrightarrow{c} UM \xrightarrow{u} \iota_{\leq -1}\tau_{\leq -1}UM \tag{2}$$

in $\mathcal{C}$ where the maps are given by the counit resp. unit of adjunction. Using that F is right t-exact and G is left t-exact we can define a morphism

$$\kappa_{\geq 0}(M)\colon F\iota_{\geq 0}\tau_{\geq 0}UM \to \iota_{\geq 0}\tau_{\geq 0}FUM \xrightarrow{\kappa(M)} \iota_{\geq 0}\tau_{\geq 0}GUM \to G\iota_{\geq 0}\tau_{\geq 0}UM$$

where the first and last morphisms are the ones that we constructed in the proof of Lemma 3.2. Equipping the left hand term of (2) with this morphism we can promote it to an object of $\mathrm{LEq}(F,G)$. Furthermore, the left hand map in (2) is a morphism in $\mathrm{LEq}(F,G)$ as can be seen from the commutative diagram



Here, the middle part of the diagram commutes because of the naturality of c and the upper and lower parts commute by Lemma 3.2.

So we set $M' \coloneqq (\iota_{\geq 0}\tau_{\geq 0}UM, \kappa_{\geq 0}(M))$ and $M'' \coloneqq \mathrm{cofib}(M' \xrightarrow{c} M)$ where the cofiber is taken in the category $\mathrm{LEq}(F,G)$. Since $\mathrm{LEq}(F,G)$ is stable this gives us a fiber sequence $M' \to M \to M''$ in $\mathrm{LEq}(F,G)$. As U is exact by Proposition 2.6(e), we have

$$UM'' \simeq \mathrm{cofib}(\iota_{\geq 0}\tau_{\geq 0}UM \xrightarrow{c} UM) \simeq \iota_{\leq -1}\tau_{\leq -1}UM$$

which shows that $M'' \in \mathrm{LEq}(F,G)_{\leq -1}$. Moreover, note that $M' \in \mathrm{LEq}(F,G)_{\geq 0}$ since $UM' = \iota_{\geq 0}\tau_{\geq 0}UM$.

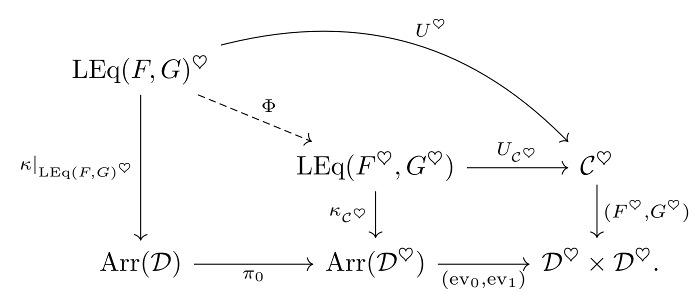
The t-exactness of the forgetful functor is clear. □

We can also compute the heart of the induced t-structure.

Proposition 3.4. *Assume that there is a t-structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ on $\mathcal{D}$ such that $F\colon \mathcal{C} \to \mathcal{D}$ is right t-exact and $G\colon \mathcal{C} \to \mathcal{D}$ is left t-exact. The heart of the t-structure $(\mathrm{LEq}(F,G)_{\geq 0}, \mathrm{LEq}(F,G)_{\leq 0})$ is given by $\mathrm{LEq}(F,G)^{\heartsuit} \simeq \mathrm{LEq}(F^{\heartsuit}, G^{\heartsuit})$ where $F^{\heartsuit}$ resp. $G^{\heartsuit}$ denotes the composition $\mathcal{C}^{\heartsuit} \hookrightarrow \mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\pi_0} \mathcal{D}^{\heartsuit}$ and for G respectively.*

Proof. The forgetful functor $U\colon \mathrm{LEq}(F,G) \to \mathcal{C}$ is t-exact by Proposition 3.3, so it restricts to a functor $U^{\heartsuit}\colon \mathrm{LEq}(F,G)^{\heartsuit} \to \mathcal{C}^{\heartsuit}$ of the hearts. Moreover, we denote by $U_{\mathcal{C}^{\heartsuit}}\colon \mathrm{LEq}(F^{\heartsuit}, G^{\heartsuit}) \to \mathcal{C}^{\heartsuit}$ the forgetful functor of $\mathrm{LEq}(F^{\heartsuit}, G^{\heartsuit})$.

We define a functor $\Phi\colon \mathrm{LEq}(F,G)^{\heartsuit} \to \mathrm{LEq}(F^{\heartsuit}, G^{\heartsuit})$ via the universal property of the pullback and the commutative diagram



We show that Φ is an equivalence by proving that it is essentially surjective and fully faithful.

For the essential surjectivity let $M \in \mathrm{LEq}(F^\heartsuit, G^\heartsuit)$. Define a morphism $\tilde{\kappa}_M \colon FU_{\mathcal{C}^\heartsuit}M \to GU_{\mathcal{C}^\heartsuit}M$ to be the composition

$$\begin{array}{ccc}
FU_{\mathcal{C}^\heartsuit}M & \xrightarrow{\quad\tilde{\kappa}_M\quad} & GU_{\mathcal{C}^\heartsuit}M \\
{\scriptstyle u}\big\downarrow & & \big\uparrow{\scriptstyle c} \\
\iota_{\leq 0}\tau_{\leq 0}FU_{\mathcal{C}^\heartsuit}M & & \iota_{\geq 0}\tau_{\geq 0}GU_{\mathcal{C}^\heartsuit}M \\
{\scriptstyle \simeq}\big\downarrow & & \big\uparrow{\scriptstyle \simeq} \\
F^\heartsuit U_{\mathcal{C}^\heartsuit}M & \xrightarrow{\kappa_{\mathcal{C}^\heartsuit}(M)} & G^\heartsuit U_{\mathcal{C}^\heartsuit}M
\end{array}$$

where the arrows u and c are the unit resp. counit of adjunction, and the equivalences exist because F is right t-exact and G is left t-exact. Equipping $U_{\mathcal{C}^\heartsuit}M$ with this morphism we can promote it to an object $(U_{\mathcal{C}^\heartsuit}M, \tilde{\kappa}_M)$ of $\mathrm{LEq}(F,G)^\heartsuit$. As $\pi_0(\tilde{\kappa}_M) \simeq \pi_0(\kappa_{\mathcal{C}^\heartsuit}(M)) \simeq \kappa_{\mathcal{C}^\heartsuit}(M)$, the functor Φ maps $(U_{\mathcal{C}^\heartsuit}M, \tilde{\kappa}_M)$ to $(U_{\mathcal{C}^\heartsuit}M, \kappa_{\mathcal{C}^\heartsuit}(M)) = M$. Thus, the functor Φ is essentially surjective.

To prove fully faithfulness of Φ, let $M, N \in \mathrm{LEq}(F,G)^\heartsuit$. We show that the map

$$\mathrm{Map}_{\mathrm{LEq}(F,G)}(M,N) \to \mathrm{Map}_{\mathrm{LEq}(F^\heartsuit,G^\heartsuit)}(\Phi M, \Phi N)$$

induced by Φ is an equivalence by using the description of the mapping spaces from Proposition 2.6(a). Recall that $\mathrm{Map}_{\mathrm{LEq}(F,G)}(M,N)$ is given by

$$\mathrm{Eq}\left(\mathrm{Map}_{\mathcal{C}}(UM, UN) \underset{\kappa(N)_*F}{\overset{\kappa(M)^*G}{\rightrightarrows}} \mathrm{Map}_{\mathcal{D}}(FUM, GUN) \right)$$

and that $\mathrm{Map}_{\mathrm{LEq}(F^\heartsuit,G^\heartsuit)}(\Phi M, \Phi N)$ is given by

$$\mathrm{Eq}\left(\mathrm{Map}_{\mathcal{C}^\heartsuit}(U^\heartsuit M, U^\heartsuit N) \underset{\pi_0(\kappa(N))_*F^\heartsuit}{\overset{\pi_0(\kappa(M))^*G^\heartsuit}{\rightrightarrows}} \mathrm{Map}_{\mathcal{D}^\heartsuit}(F^\heartsuit U^\heartsuit M, G^\heartsuit U^\heartsuit N) \right).$$

There is a commutative diagram

$$\begin{array}{ccc}
\mathrm{Map}_{\mathcal{C}}(UM, UN) & \xrightarrow{\quad\simeq\quad} & \mathrm{Map}_{\mathcal{C}^\heartsuit}(U^\heartsuit M, U^\heartsuit N) \\
{\scriptstyle G}\big\downarrow & & \big\downarrow{\scriptstyle G^\heartsuit} \\
\mathrm{Map}_{\mathcal{D}}(GUM, GUN) & \xrightarrow{\pi_0} & \mathrm{Map}_{\mathcal{D}^\heartsuit}(G^\heartsuit U^\heartsuit M, G^\heartsuit U^\heartsuit N) \\
{\scriptstyle \kappa(M)^*}\big\downarrow & & \big\downarrow{\scriptstyle \pi_0(\kappa(M))^*} \\
\mathrm{Map}_{\mathcal{D}}(FUM, GUN) & \xrightarrow{\pi_0} & \mathrm{Map}_{\mathcal{D}^\heartsuit}(F^\heartsuit U^\heartsuit M, G^\heartsuit U^\heartsuit N).
\end{array}$$

A similar diagram can be drawn for the map $\kappa(N)_*F$. Putting both diagrams together we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(UM,UN) & \underset{\kappa(N)_*F}{\overset{\kappa(M)^*G}{\rightrightarrows}} & \mathrm{Map}_{\mathcal{D}}(FUM,GUN) \\ \downarrow \simeq & & \downarrow \pi_0 \\ \mathrm{Map}_{\mathcal{C}^\heartsuit}(U^\heartsuit M,U^\heartsuit N) & \underset{\pi_0(\kappa(N))_*F^\heartsuit}{\overset{\pi_0(\kappa(M))^*G^\heartsuit}{\rightrightarrows}} & \mathrm{Map}_{\mathcal{D}^\heartsuit}(F^\heartsuit U^\heartsuit M,G^\heartsuit U^\heartsuit N). \end{array}$$

Taking the equalizer of the rows, the vertical maps in the diagram induce the map Φ. Thus, it suffices to show that the vertical arrow on the right hand side is an equivalence. This is the case as it is given by the chain of equivalences

$$\begin{aligned} \mathrm{Map}_{\mathcal{D}}(FUM,GUN) &\simeq \mathrm{Map}_{\mathcal{D}}(\iota_{\leq 0}\tau_{\leq 0}FUM,GUN) \\ &\simeq \mathrm{Map}_{\mathcal{D}}(\iota_{\leq 0}\tau_{\leq 0}FUM,\iota_{\geq 0}\tau_{\geq 0}GUN) \\ &\simeq \mathrm{Map}_{\mathcal{D}^\heartsuit}(F^\heartsuit U^\heartsuit M,G^\heartsuit U^\heartsuit N) \end{aligned}$$

where we used the left t-exactness of G and the adjunction $\tau_{\leq 0} \dashv \iota_{\leq 0}$ in the first equivalence, the right t-exactness of F and the adjunction $\iota_{\geq 0} \dashv \tau_{\geq 0}$ in the second equivalence and both t-exactness properties in the last equivalence. □

From the definition of the t-structure and the description of its heart we immediately get.

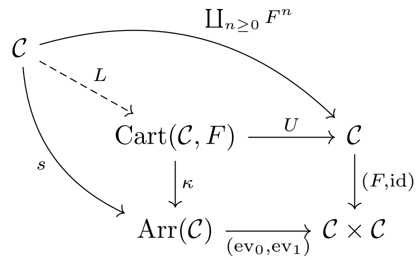
Corollary 3.5. *Assume that there is a t-structure $(\mathcal{D}_{\geq 0},\mathcal{D}_{\leq 0})$ on $\mathcal{D}$ such that $F\colon \mathcal{C}\to\mathcal{D}$ is right t-exact and $G\colon \mathcal{C}\to\mathcal{D}$ is left t-exact. The forgetful functor U is t-exact and there is a cartesian diagram*

$$\begin{array}{ccc} \mathrm{LEq}(F^\heartsuit,G^\heartsuit) & \xrightarrow{U_{\mathcal{C}^\heartsuit}} & \mathcal{C}^\heartsuit \\ \downarrow & & \downarrow \\ \mathrm{LEq}(F,G) & \xrightarrow{U} & \mathcal{C}. \end{array}$$

4 The left adjoint of the forgetful functor

Let $\mathcal{C}$ be an ∞-category that admits countable coproducts and let $F\colon \mathcal{C}\to\mathcal{C}$ be an endofunctor that preserves them. In this section we define the left adjoint of the forgetful functor $U\colon \mathrm{Cart}(\mathcal{C},F)\to\mathcal{C}$. It follows that U exhibits $\mathrm{Cart}(\mathcal{C},F)$ as monadic over $\mathcal{C}$, cf. [Lur17a, Definition 4.7.3.4]. This is the first key input for the proof of our main theorem. We conclude the section by showing that the left adjoint of U is t-exact if $\mathcal{C}$ is a stable ∞-category equipped with a t-structure that is compatible with coproducts (cf. [Lur17a, Definition 1.2.2.12]).

Definition 4.1. Let $\mathcal{C}$ be an ∞-category that admits all countable coproducts and let $F\colon \mathcal{C}\to\mathcal{C}$ be an endofunctor that preserves countable coproducts. We define a functor $L\colon \mathcal{C}\to\mathrm{Cart}(\mathcal{C},F)$ via the universal property of the pullback and the diagram



where $\coprod_{n\geq 0} F^n$ is the coproduct in $\mathrm{Fun}(\mathcal{C},\mathcal{C})$ of the functors $F^n\colon \mathcal{C}\to\mathcal{C}$ and the functor s is defined as follows:

Note that specifying a functor $\mathcal{C}\to\mathrm{Arr}(\mathcal{C})$ is the same as giving an arrow in $\mathrm{Fun}(\mathcal{C},\mathcal{C})$, i.e. a natural transformation. Under this equivalence the functor s corresponds to the natural transformation

$$\coprod_{n\geq 0}\mathrm{inc}_{n+1}\colon \coprod_{n\geq 0} F^{n+1}\to\coprod_{k\geq 0} F^k$$

where $\mathrm{inc}_{n+1}\colon F^{n+1}\to\coprod_{k\geq 0}F^k$ denotes the natural map into the coproduct.

As F preserves countable coproducts, the outer diagram commutes and hence defines a functor L.

Proposition 4.2. *Let $\mathcal{C}$ be an ∞-category that admits all countable coproducts and let $F\colon\mathcal{C}\to\mathcal{C}$ be an endofunctor that preserves countable coproducts. Then the functor L is left adjoint to the forgetful functor $U\colon\ \mathrm{Cart}(\mathcal{C},F)\to\mathcal{C}$.*

Proof. First, we define the unit map $u\colon\ \mathrm{id}_{\mathcal{C}}\to UL$ to be the inclusion of the zeroth component $\mathrm{inc}_0\colon\ \mathrm{id}_{\mathcal{C}}=F^0\to\coprod_{n\geq 0}F^n$. By [Lur17b, Proposition 5.2.2.8] it is enough to show that the map

$$\mathrm{Map}_{\mathrm{Cart}(\mathcal{C},F)}(LX,M)\xrightarrow{U}\mathrm{Map}_{\mathcal{C}}(ULX,UM)\xrightarrow{u^*}\mathrm{Map}_{\mathcal{C}}(X,UM)\qquad(3)$$

is an equivalence for all objects $X\in\mathcal{C}$ and $M\in\mathrm{Cart}(\mathcal{C},F)$.

Note that there are equivalences

$$\begin{aligned}\mathrm{Map}_{\mathrm{Cart}(\mathcal{C},F)}(LX,M)&\simeq\mathrm{Eq}\left(\mathrm{Map}_{\mathcal{C}}(ULX,UM)\underset{\kappa(M)_*F}{\overset{\kappa(LX)^*}{\rightrightarrows}}\mathrm{Map}_{\mathcal{C}}(FULX,UM)\right)\\&\simeq\mathrm{Eq}\left(\mathrm{Map}_{\mathcal{C}}\left(\coprod_{n\geq 0}F^nX,UM\right)\rightrightarrows\mathrm{Map}_{\mathcal{C}}\left(\coprod_{n\geq 0}F^{n+1}X,UM\right)\right)\\&\simeq\mathrm{Eq}\left(\prod_{n\geq 0}\mathrm{Map}_{\mathcal{C}}(F^nX,UM)\underset{\Psi}{\overset{\Phi}{\rightrightarrows}}\prod_{n\geq 0}\mathrm{Map}_{\mathcal{C}}(F^{n+1}X,UM)\right),\end{aligned}$$

where we used Corollary 2.7(a) in the first equivalence, the definition of L and the assumption that F preserves countable coproducts in the second equivalence and the universal property of coproducts in the last equivalence. Using the definition of L we see that the map Φ is defined by $\mathrm{pr}_n \Phi \coloneqq \mathrm{pr}_{n+1}$ where

$$\mathrm{pr}_n \colon \prod_{k\geq 0} \mathrm{Map}_{\mathcal{C}}(F^k X, UM) \to \mathrm{Map}_{\mathcal{C}}(F^n X, UM)$$

denotes the projection on the n-th factor. Moreover, the map Ψ is defined by $\mathrm{pr}_n \Psi \coloneqq \kappa(M)_* F \,\mathrm{pr}_n$. Thus, the composition (3) is equivalent to the composition

$$\mathrm{Eq}(\Phi, \Psi) \xrightarrow{i} \prod_{n\geq 0} \mathrm{Map}_{\mathcal{C}}(F^n X, UM) \xrightarrow{\mathrm{pr}_0} \mathrm{Map}_{\mathcal{C}}(X, UM)\,, \tag{4}$$

where i is the natural map out of the equalizer.

To show that this composition is an equivalence we construct a map

$$\beta \colon \mathrm{Map}_{\mathcal{C}}(X, UM) \to \mathrm{Eq}(\Phi, \Psi)$$

that satisfies $\mathrm{pr}_0 \, i\beta \simeq \mathrm{id}$ and $\beta \,\mathrm{pr}_0 \, i \simeq \mathrm{id}$ and hence is an inverse of (4). For that, we use the universal property of the equalizer and construct a map

$$\alpha \colon \mathrm{Map}_{\mathcal{C}}(X, UM) \to \prod_{n\geq 0} \mathrm{Map}_{\mathcal{C}}(F^n X, UM)$$

with a homotopy $\Phi\alpha \simeq \Psi\alpha$. We define inductively

$$\mathrm{pr}_0 \, \alpha \coloneqq \mathrm{id} \colon \mathrm{Map}_{\mathcal{C}}(X, UM) \to \mathrm{Map}_{\mathcal{C}}(F^0 X, UM)$$

and

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(X, UM) & \xrightarrow{\mathrm{pr}_{n+1}\,\alpha} & \mathrm{Map}_{\mathcal{C}}(F^{n+1} X, UM) \\ {\scriptstyle \mathrm{pr}_n\,\alpha}\downarrow & & \uparrow{\scriptstyle \kappa(M)_*} \\ \mathrm{Map}_{\mathcal{C}}(F^n X, UM) & \xrightarrow[F]{} & \mathrm{Map}_{\mathcal{C}}(F^{n+1} X, FUM). \end{array}$$

This induces a map $\alpha \colon \mathrm{Map}_{\mathcal{C}}(X, UM) \to \prod_{n\geq 0} \mathrm{Map}_{\mathcal{C}}(F^n X, UM)$ and

$$\Phi\alpha = (\mathrm{pr}_{n+1}\,\alpha)_n \simeq (\kappa(M)_* F \,\mathrm{pr}_n\,\alpha)_n = \Psi\alpha$$

holds by construction. Therefore, we get a map $\beta \colon \mathrm{Map}_{\mathcal{C}}(X, UM) \to \mathrm{Eq}(\Phi, \Psi)$ that satisfies $i\beta \simeq \alpha$.

It follows immediately from the construction of α that we have

$$\mathrm{pr}_0 \, i\beta \simeq \mathrm{pr}_0 \, \alpha \simeq \mathrm{id}\ .$$

Thus, it remains to show that $\beta \,\mathrm{pr}_0 \, i \simeq \mathrm{id}$. For that, it suffices to show that for all n we have $\mathrm{pr}_n \, i\beta \,\mathrm{pr}_0 \, i \simeq \mathrm{pr}_n \, i$. We do this inductively. For $n = 0$ we have

$$\mathrm{pr}_0 \, i\beta \,\mathrm{pr}_0 \, i \simeq \mathrm{id}\,\mathrm{pr}_0 \, i \simeq \mathrm{pr}_0 \, i$$

by the above. For $n > 0$ we compute

$$\begin{aligned}\mathrm{pr}_{n+1} i\beta \,\mathrm{pr}_0\, i &\simeq \mathrm{pr}_{n+1} \alpha \,\mathrm{pr}_0\, i\\ &\simeq \kappa(M)_* F \,\mathrm{pr}_n \alpha \,\mathrm{pr}_0\, i\\ &\simeq \kappa(M)_* F \,\mathrm{pr}_n\, i\\ &\simeq \mathrm{pr}_{n+1} i\end{aligned}$$

where we used the definition of β in the first equivalence, the definition of α in the second equivalence, the induction hypothesis in the third equivalence and the definition of Φ and Ψ in the last equivalence. □

Corollary 4.3. *Let $\mathcal{C}$ be an ∞-category that admits all countable coproducts and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor that preserves countable coproducts. Let $G\colon \mathcal{C} \to \mathcal{D}$ be a functor that admits a left adjoint $G^L\colon \mathcal{D} \to \mathcal{C}$. Then the forgetful functor $U\colon \mathrm{LEq}(F, G) \to \mathcal{C}$ admits a left adjoint. It can be described as in Definition 4.1 by replacing F by $G^L F$.*

Proof. There is an equivalence of ∞-categories $\mathrm{LEq}(F, G) \simeq \mathrm{Cart}(\mathcal{C}, G^L F)$ by Proposition 2.9. Note that the functor G^L preserves countable coproducts as it is a left adjoint. Hence, we can apply Proposition 4.2. □

Corollary 4.4. *Let $\mathcal{C}$ be an ∞-category that admits all countable coproducts and let $G\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor that admits a left adjoint G^L. Then the forgetful functor $U\colon \mathrm{Frob}(\mathcal{C}, G) \to \mathcal{C}$ admits a left adjoint. It can be described as in Definition 4.1 by replacing F by G^L.*

Proof. Apply Corollary 4.3 with $F = \mathrm{id}_{\mathcal{C}}$. □

As a consequence, we obtain that the forgetful functor is monadic. The proof of this is an application of the ∞-categorical Barr-Beck Theorem [Lur17a, Theorem 4.7.3.5]. To see this, we first show the following.

Lemma 4.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors. The ∞-categories $\mathrm{Cart}(\mathcal{C}, F)$, $\mathrm{Frob}(\mathcal{C}, G)$ (in the case $\mathcal{D} = \mathcal{C}$) and $\mathrm{LEq}(F, G)$ admit colimits of U-split simplicial objects (cf. [Lur17a, Definition 4.7.2.2]) and U preserves them.*

Proof. Let $X_\bullet\colon N(\Delta_+^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{Cart}(\mathcal{C}, F)$ be a U-split simplicial object, i.e. $UX_\bullet$ is a split simplicial object in $\mathcal{C}$. By [Lur17b, Lemma 6.1.3.16], $UX_\bullet$ admits a colimit in $\mathcal{C}$. Note that the functor F preserves this colimit by [Lur17a, Remark 4.7.2.4]. Thus, the claim about $\mathrm{Cart}(\mathcal{C}, F)$ follows from Corollary 2.7(d).

To prove the claims about $\mathrm{Frob}(\mathcal{C}, G)$ and $\mathrm{LEq}(F, G)$ we can do an analogous argument using Corollary 2.8(d), resp. Proposition 2.6(d) at the end. □

Theorem 4.6. *(a) Let $\mathcal{C}$ be an ∞-category and let $F\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor such that the forgetful functor $U\colon \mathrm{Cart}(\mathcal{C}, F) \to \mathcal{C}$ admits a left adjoint (e.g. if $\mathcal{C}$ admits countable coproducts and F preserves them). Then the functor U exhibits $\mathrm{Cart}(\mathcal{C}, F)$ as monadic over $\mathcal{C}$ (in the sense of [Lur17a, Definition 4.7.3.4]).*

(b) *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors such that the forgetful functor $U\colon \mathrm{LEq}(F,G) \to \mathcal{C}$ admits a left adjoint. Then the functor U exhibits $\mathrm{LEq}(F,G)$ as monadic over $\mathcal{C}$.*

(c) *Let $\mathcal{C}$ be an ∞-category and let $G\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor such that the forgetful functor $U\colon \mathrm{Frob}(\mathcal{C},G) \to \mathcal{C}$ admits a left adjoint. Then the functor U exhibits $\mathrm{Frob}(\mathcal{C},G)$ as monadic over $\mathcal{C}$.*

Proof. By the ∞-categorical Barr-Beck Theorem [Lur17a, Theorem 4.7.3.5] it is enough to show that U is conservative, $\mathrm{Cart}(\mathcal{C},F)$ (resp. $\mathrm{LEq}(F,G)$ or $\mathrm{Frob}(\mathcal{C},G)$) admits colimits of U-split simplicial objects and U preserves these colimits. The latter holds by Lemma 4.5. Moreover, we have already seen that U is conservative in Corollary 2.7(b) (resp. Proposition 2.6(b) or Corollary 2.8(b)). □

We finish this section by investigating the t-exactness properties of the functor L. Under the additional assumption that the t-structure on $\mathcal{C}$ is compatible with coproducts we get an analog of Corollary 3.5 which is an important ingredient for the proof of our main theorem.

Lemma 4.7. *Let $\mathcal{C}$ be a stable ∞-category with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Suppose that $\mathcal{C}_{\leq 0}$ is stable under coproducts.*

(a) *Let $F\colon \mathcal{C} \to \mathcal{C}$ be a t-exact endofunctor that preserves countable coproducts. Then the left adjoint of the forgetful functor $U\colon \mathrm{Cart}(\mathcal{C},F) \to \mathcal{C}$ from Definition 4.1 is t-exact (with respect to the induced t-structure on $\mathrm{Cart}(\mathcal{C},F)$ described in Proposition 3.3).*

(b) *Let $\mathcal{D}$ be a stable ∞-category with a t-structure and $F\colon \mathcal{C} \to \mathcal{D}$ be a t-exact functor that preserves countable coproducts. Let $G\colon \mathcal{C} \to \mathcal{D}$ be a functor that admits a t-exact left adjoint. Then the left adjoint of the forgetful functor $U\colon \mathrm{LEq}(F,G) \to \mathcal{C}$ is t-exact.*

(c) *Let $G\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor that admits a t-exact left adjoint. Then the left adjoint of the forgetful functor $U\colon \mathrm{Frob}(\mathcal{C},G) \to \mathcal{C}$ is t-exact.*

Proof. Note for (b) and (c) that G is left t-exact by [BBD82, Proposition 1.3.17(iii)] because it is right adjoint to a t-exact functor. In particular, the ∞-categories $\mathrm{LEq}(F,G)$ and $\mathrm{Frob}(\mathcal{C},G)$ carry the induced t-structure described in Proposition 3.3.

In all three cases we denote the left adjoint of the forgetful functor by L. By Proposition 3.3 we have to show that $UL(\mathcal{C}_{\geq 0}) \subseteq \mathcal{C}_{\geq 0}$ and that $UL(\mathcal{C}_{\leq 0}) \subseteq \mathcal{C}_{\leq 0}$. This follows directly from the definition of L (Definition 4.1, Corollary 4.3, Corollary 4.4), the assumptions on F and G and the fact that $C_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ are stable under coproducts. □

Corollary 4.8. *Under the respective assumptions of the previous lemma there are commutative diagrams*

(a)

$$\begin{array}{ccc} \mathcal{C}^{\heartsuit} & \xrightarrow{L_{\mathcal{C}^{\heartsuit}}} & \mathrm{Cart}(\mathcal{C}^{\heartsuit}, F^{\heartsuit}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{C} & \xrightarrow{L_{\mathcal{C}}} & \mathrm{Cart}(\mathcal{C}, F) \end{array}$$

(b)

$$\begin{array}{ccc} \mathcal{C}^{\heartsuit} & \xrightarrow{L_{\mathcal{C}^{\heartsuit}}} & \mathrm{LEq}(F^{\heartsuit}, G^{\heartsuit}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{C} & \xrightarrow{L_{\mathcal{C}}} & \mathrm{LEq}(F, G) \end{array}$$

(c)

$$\begin{array}{ccc} \mathcal{C}^{\heartsuit} & \xrightarrow{L_{\mathcal{C}^{\heartsuit}}} & \mathrm{Frob}(\mathcal{C}^{\heartsuit}, G^{\heartsuit}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{C} & \xrightarrow{L_{\mathcal{C}}} & \mathrm{Frob}(\mathcal{C}, G) \end{array}$$

In each case the horizontal functors denote the left adjoint of the forgetful functor.

Proof. We only prove (a), an analogous proof shows (b) and (c). For the sake of clarity, we add an index to L and U to indicate to which category they refer.

As $L_{\mathcal{C}}$ is t-exact by Lemma 4.7, it restricts to a functor

$$L_{\mathcal{C}}^{\heartsuit} \colon \mathcal{C}^{\heartsuit} \to \mathrm{Cart}(\mathcal{C}, F)^{\heartsuit} .$$

The same is true for the forgetful functor $U_{\mathcal{C}} \colon \mathrm{Cart}(\mathcal{C}, F) \to \mathcal{C}$ by Corollary 3.5. The restricted functors $L_{\mathcal{C}}^{\heartsuit}$ and $U_{\mathcal{C}}^{\heartsuit}$ are adjoint by [BBD82, Proposition 1.3.17(iii)] because $L_{\mathcal{C}}$ is left adjoint to $U_{\mathcal{C}}$. But again by Corollary 3.5, $U_{\mathcal{C}}^{\heartsuit} \simeq U_{\mathcal{C}^{\heartsuit}}$, so $L_{\mathcal{C}}^{\heartsuit}$ and $L_{\mathcal{C}^{\heartsuit}}$ are left adjoints of equivalent functors. Hence, $L_{\mathcal{C}}^{\heartsuit}$ and $L_{\mathcal{C}^{\heartsuit}}$ are equivalent (cf. [Lur17b, Remark 5.2.2.2]). □

5 The derived ∞-category of Cartier modules

In this section we state and prove the main theorem of this paper. The proof relies on a consequence of the ∞-categorical Barr-Beck Theorem [Lur17a, Theorem 4.7.3.5] which we recall in Proposition 5.3.

Recall that a Grothendieck abelian category is a presentable abelian category that has exact filtered colimits. In particular, Grothendieck abelian categories satisfy Grothendieck's axioms AB3, AB4 and AB5, cf. [Sta24, Tag 079A].

Furthermore, if $\mathcal{A}$ is a Grothendieck abelian category we can define its derived ∞-category $\mathcal{D}(\mathcal{A})$, cf. [Lur17a, Definition 1.3.5.8]. The derived ∞-category carries a natural t-structure (cf. [Lur17a, Definition 1.3.5.16]) whose heart is given by $\mathcal{D}(\mathcal{A})^{\heartsuit} \simeq \mathcal{A}$ (cf. [Lur18, Remark C.5.4.11]).

For $G\colon \mathcal{A} \to \mathcal{B}$ a colimit-preserving exact functor between Grothendieck abelian categories there is an induced colimit-preserving t-exact functor

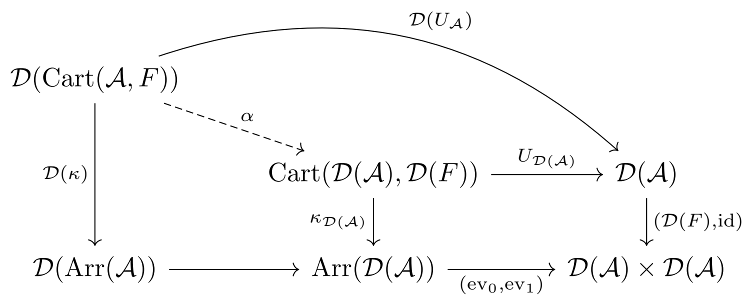
$$\mathcal{D}(G)\colon \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$$

by Proposition A.2. Moreover, we can also apply Proposition A.2 to the equivalence

$$\operatorname{Arr}(\mathcal{A}) \xrightarrow{\simeq} \operatorname{Arr}\left(\mathcal{D}(\mathcal{A})^{\heartsuit}\right) \xrightarrow{\simeq} (\operatorname{Arr}(\mathcal{D}(\mathcal{A})))^{\heartsuit}$$

to get a functor $\mathcal{D}(\operatorname{Arr}(\mathcal{A})) \to \operatorname{Arr}(\mathcal{D}(\mathcal{A}))$.

Theorem 5.1. *Let $\mathcal{A}$ be a Grothendieck abelian category and $F\colon \mathcal{A} \to \mathcal{A}$ be an exact and colimit-preserving functor. Then $\operatorname{Cart}(\mathcal{A}, F)$ is a Grothendieck abelian category and the natural functor $\alpha\colon \mathcal{D}(\operatorname{Cart}(\mathcal{A}, F)) \to \operatorname{Cart}(\mathcal{D}(\mathcal{A}), \mathcal{D}(F))$ given by the universal property of the pullback and the diagram*



is an equivalence of ∞-categories.

Furthermore, the functor α is t-exact with respect to the usual t-structure on $\mathcal{D}(\operatorname{Cart}(\mathcal{A}, F))$ and the induced t-structure on $\operatorname{Cart}(\mathcal{D}(\mathcal{A}), \mathcal{D}(F))$ described in Proposition 3.3.

Note that the outer diagram in the theorem indeed commutes as this can be checked on hearts by Proposition A.2 where the bottom composition is given by $(\mathrm{ev}_0, \mathrm{ev}_1)$. So, it boils down to the commutative diagram defining $\operatorname{Cart}(\mathcal{A}, F)$.

Remark 5.2. *By Proposition A.2, the equivalence of the heart*

$$\operatorname{Cart}(\mathcal{A}, F) \simeq \operatorname{Cart}(\mathcal{D}(\mathcal{A}), \mathcal{D}(F))^{\heartsuit}$$

described in Proposition 3.4 also induces a functor

$$\mathcal{D}(\operatorname{Cart}(\mathcal{A}, F)) \to \operatorname{Cart}(\mathcal{D}(\mathcal{A}), \mathcal{D}(F))\,.$$

It can be shown that this functor is equivalent to the functor α of Theorem 5.1 by checking it on the heart and applying Proposition A.2.

To prove the theorem we use the following result which is a consequence of the ∞-categorical Barr-Beck Theorem [Lur17a, Theorem 4.7.3.5].

Proposition 5.3. *Suppose we are given a commutative diagram of ∞-categories*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\alpha} & \mathcal{C}' \\ & {\scriptstyle U}\searrow \quad \swarrow{\scriptstyle U'} & \\ & \mathcal{D} & \end{array}$$

and assume the following:

(1) The functor U exhibits $\mathcal{C}$ as monadic over $\mathcal{D}$.

(2) The functor U' exhibits $\mathcal{C}'$ as monadic over $\mathcal{D}$.

(3) Denote the left adjoints of U and U' by L and L', respectively. For each object $D \in \mathcal{D}$, the unit map $D \to UL(D) \simeq U'\alpha L(D)$ induces an equivalence $L'(D) \to \alpha L(D)$ in $\mathcal{C}'$.

Then α is an equivalence.

Proof. This is a reformulation of [Lur17a, Corollary 4.7.3.16] using [Lur17a, Theorem 4.7.3.5]. □

So, to prove Theorem 5.1 we show that for the diagram

$$\begin{array}{ccc} \mathcal{D}(\mathrm{Cart}(\mathcal{A}, F)) & \xrightarrow{\alpha} & \mathrm{Cart}(\mathcal{D}(\mathcal{A}), \mathcal{D}(F)) \\ & {\scriptstyle \mathcal{D}(U_{\mathcal{A}})}\searrow \quad \swarrow{\scriptstyle U_{\mathcal{D}(\mathcal{A})}} & \\ & \mathcal{D}(\mathcal{A}) & \end{array}$$

the assumptions (1) to (3) of Proposition 5.3 are satisfied where we keep the notation from Theorem 5.1. Note that the category $\mathrm{Cart}(\mathcal{A}, F)$ is Grothendieck abelian by Corollary 2.7(i), so that it makes sense to write $\mathcal{D}(\mathrm{Cart}(\mathcal{A}, F))$. We start with the proof of (1).

Proposition 5.4. *Let $\mathcal{A}$ be a Grothendieck abelian category and $F\colon \mathcal{A} \to \mathcal{A}$ be an exact and colimit-preserving functor. Then the functor*

$$\mathcal{D}(U_{\mathcal{A}})\colon \mathcal{D}(\mathrm{Cart}(\mathcal{A}, F)) \to \mathcal{D}(\mathcal{A})$$

exhibits $\mathcal{D}(\mathrm{Cart}(\mathcal{A}, F))$ as monadic over $\mathcal{D}(\mathcal{A})$.

Proof. By Proposition 4.2 the functor $U_{\mathcal{A}}\colon \mathrm{Cart}(\mathcal{A}, F) \to \mathcal{A}$ admits a left adjoint $L_{\mathcal{A}}$. Using Theorem 4.6(a) we see that $U_{\mathcal{A}}$ exhibits $\mathrm{Cart}(\mathcal{A}, F)$ as monadic over $\mathcal{A}$. Noting that coproducts in Grothendieck abelian categories are exact by axiom AB4, we get that $L_{\mathcal{A}}$ is an exact functor. Thus, we conclude by applying Corollary A.8. □

Proof of Theorem 5.1. First note that the category $\mathrm{Cart}(\mathcal{A}, F)$ is Grothendieck abelian by Corollary 2.7(i).

Moreover, the t-exactness of the functor α follows immediately from the definition of the induced t-structure on $\mathrm{Cart}(\mathcal{D}(\mathcal{A}), \mathcal{D}(F))$ and the t-exactness of $\mathcal{D}(U_{\mathcal{A}}) \simeq U_{\mathcal{D}(\mathcal{A})}\alpha$.

So, it remains to show that the natural functor

$$\alpha\colon \mathcal{D}(\mathrm{Cart}(\mathcal{A}, F)) \to \mathrm{Cart}(\mathcal{D}(\mathcal{A}), \mathcal{D}(F))$$

is an equivalence. As we already described above the proof proceeds by considering the commutative diagram

$$\begin{array}{ccc} \mathcal{D}(\mathrm{Cart}(\mathcal{A}, F)) & \xrightarrow{\alpha} & \mathrm{Cart}(\mathcal{D}(\mathcal{A}), \mathcal{D}(F)) \\ & {\scriptstyle \mathcal{D}(U_{\mathcal{A}})} \searrow \quad \swarrow {\scriptstyle U_{\mathcal{D}(\mathcal{A})}} & \\ & \mathcal{D}(\mathcal{A}) & \end{array}$$

and checking that the assumptions (1) to (3) of Proposition 5.3 are satisfied.

(1) is Proposition 5.4.

For (2) note that $\mathcal{D}(\mathcal{A})$ is presentable by [Lur17a, Proposition 1.3.5.21(1)] and that $\mathcal{D}(F)$ preserves colimits by definition. Hence, we can apply Theorem 4.6(a) to get (2).

It remains to prove (3). We have to show that for each object $M \in \mathcal{D}(\mathcal{A})$ there is an equivalence $L_{\mathcal{D}(\mathcal{A})}(M) \to \alpha\mathcal{D}(L_{\mathcal{A}})(M)$ induced by the unit of the adjunction $\mathcal{D}(L_{\mathcal{A}}) \dashv \mathcal{D}(U_{\mathcal{A}})$. As $U_{\mathcal{D}(\mathcal{A})}$ is conservative it suffices to show that the map

$$U_{\mathcal{D}(\mathcal{A})}L_{\mathcal{D}(\mathcal{A})}(M) \to U_{\mathcal{D}(\mathcal{A})}\alpha\mathcal{D}(L_{\mathcal{A}})(M) \simeq \mathcal{D}(U_{\mathcal{A}})\mathcal{D}(L_{\mathcal{A}})(M)$$

is an equivalence.

The functor $U_{\mathcal{D}(\mathcal{A})}$ is t-exact by Corollary 3.5 and colimit-preserving by Corollary 2.7(f). Note that the t-structure on $\mathcal{D}(\mathcal{A})$ is compatible with filtered colimits by [Lur17a, Proposition 1.3.5.21(3)], and hence, the functor $L_{\mathcal{D}(\mathcal{A})}$ is t-exact by Lemma 4.7(a). The functor $L_{\mathcal{D}(\mathcal{A})}$ also preserves colimits as it is a left adjoint. Moreover, the functors $\mathcal{D}(U_{\mathcal{A}})$ and $\mathcal{D}(L_{\mathcal{A}})$ are t-exact and colimit-preserving by definition. Thus, we can apply Proposition A.2 to reduce to the case $M \in \mathcal{D}(\mathcal{A})^{\heartsuit}$.

Denote by $H\colon \mathcal{D}(\mathcal{A})^{\heartsuit} \to \mathcal{D}(\mathcal{A})$ the inclusion of the heart. Then there is a commutative diagram

$$\begin{array}{ccc} \pi_0 U_{\mathcal{D}(\mathcal{A})}L_{\mathcal{D}(\mathcal{A})}H(M) & \longrightarrow & \pi_0\mathcal{D}(U_{\mathcal{A}})\mathcal{D}(L_{\mathcal{A}})H(M) \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ U_{\mathcal{A}}L_{\mathcal{A}}(M) & \xrightarrow{\mathrm{id}} & U_{\mathcal{A}}L_{\mathcal{A}}(M) \end{array}$$

where the left vertical arrow is an equivalence because of Corollary 4.8(a), Corollary 3.5 and the fact that $\pi_0 H \simeq \mathrm{id}$ and the right vertical arrow is an equivalence by definition of $\mathcal{D}(U_{\mathcal{A}})$ and $\mathcal{D}(L_{\mathcal{A}})$. The commutativity follows because the top horizontal map is induced by the composition

$$\begin{aligned} U_{\mathcal{D}(\mathcal{A})}L_{\mathcal{D}(\mathcal{A})} &\xrightarrow{\mathcal{D}(u)} U_{\mathcal{D}(\mathcal{A})}L_{\mathcal{D}(\mathcal{A})}\mathcal{D}(U_{\mathcal{A}})\mathcal{D}(L_{\mathcal{A}}) \\ &\xrightarrow{\simeq} U_{\mathcal{D}(\mathcal{A})}L_{\mathcal{D}(\mathcal{A})}U_{\mathcal{D}(\mathcal{A})}\alpha\mathcal{D}(L_{\mathcal{A}}) \\ &\xrightarrow{c_{\mathcal{D}(\mathcal{A})}} U_{\mathcal{D}(\mathcal{A})}\alpha\mathcal{D}(L_{\mathcal{A}}) \\ &\xrightarrow{\simeq} \mathcal{D}(U_{\mathcal{A}})\mathcal{D}(L_{\mathcal{A}}) \end{aligned}$$

where $\mathcal{D}(u)$ denotes the unit of the adjunction $\mathcal{D}(L_{\mathcal{A}}) \dashv \mathcal{D}(U_{\mathcal{A}})$ and $c_{\mathcal{D}(\mathcal{A})}$ the counit of the adjunction $L_{\mathcal{D}(\mathcal{A})} \dashv U_{\mathcal{D}(\mathcal{A})}$. When restricted to the heart these two adjunctions both become equivalent to the adjunction $L_{\mathcal{A}} \dashv U_{\mathcal{A}}$ and the map comes down to

$$U_{\mathcal{A}}L_{\mathcal{A}} \xrightarrow{u} U_{\mathcal{A}}L_{\mathcal{A}}U_{\mathcal{A}}L_{\mathcal{A}} \xrightarrow{c} U_{\mathcal{A}}L_{\mathcal{A}}$$

which is equivalent to the identity because of one of the triangle identities.

The diagram shows that the map

$$\pi_0 U_{\mathcal{D}(\mathcal{A})}L_{\mathcal{D}(\mathcal{A})}H(M) \to \pi_0\mathcal{D}(U_{\mathcal{A}})\mathcal{D}(L_{\mathcal{A}})H(M)$$

is an equivalence for $M \in \mathcal{D}(\mathcal{A})^{\heartsuit}$. As we explained above, this implies that the map

$$U_{\mathcal{D}(\mathcal{A})}L_{\mathcal{D}(\mathcal{A})}(M) \to \mathcal{D}(U_{\mathcal{A}})\mathcal{D}(L_{\mathcal{A}})(M)$$

is an equivalence for all $M \in \mathcal{D}(\mathcal{A})$. This completes the proof of (3) and thus also the proof of the theorem. □

Corollary 5.5. *Let $\mathcal{A}$ be a Grothendieck abelian category and $F, G\colon \mathcal{A} \to \mathcal{A}$ be exact and colimit-preserving functors. Assume that G has an exact left adjoint G^L. Then* $\mathrm{LEq}(F, G)$ *is a Grothendieck abelian category and the natural functor*

$$\alpha\colon \mathcal{D}(\mathrm{LEq}(F, G)) \to \mathrm{LEq}(\mathcal{D}(F), \mathcal{D}(G))\,,$$

defined analogously to the one in Theorem 5.1, is a t-exact equivalence of ∞-categories.

Proof. Note that by Lemma A.6, $\mathcal{D}(G^L)$ is left adjoint to $\mathcal{D}(G)$. So, by Proposition 2.9 there are equivalences $\mathrm{LEq}(F, G) \simeq \mathrm{Cart}(\mathcal{A}, G^L F)$ and

$$\mathrm{LEq}(\mathcal{D}(F), \mathcal{D}(G)) \simeq \mathrm{Cart}(\mathcal{D}(\mathcal{A}), \mathcal{D}(G^L)\mathcal{D}(F))$$

of ∞-categories. Now we can apply Theorem 5.1 to conclude. □

Corollary 5.6. *Let $\mathcal{A}$ be a Grothendieck abelian category and $G\colon \mathcal{A} \to \mathcal{A}$ be an exact and colimit-preserving functor. Assume that G has an exact left adjoint. Then* $\mathrm{Frob}(\mathcal{A}, G)$ *is a Grothendieck abelian category and the natural functor*

$$\alpha\colon \mathcal{D}(\mathrm{Frob}(\mathcal{A}, G)) \to \mathrm{Frob}(\mathcal{D}(\mathcal{A}), \mathcal{D}(G))\,,$$

defined analogously to the one in Theorem 5.1, is a t-exact equivalence of ∞-categories.

Proof. Apply Corollary 5.5 with $F = \mathrm{id}_{\mathcal{A}}$. $\square$

If we apply Theorem 5.1 with $F = \mathrm{id}_{\mathcal{A}}$ we recover the following well-known result about the category $\mathrm{End}(\mathcal{A})$ of endomorphisms of $\mathcal{A}$.

Corollary 5.7. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then there is a canonical equivalence $\mathcal{D}(\mathrm{End}(\mathcal{A})) \xrightarrow{\simeq} \mathrm{End}(\mathcal{D}(\mathcal{A}))$ of ∞-categories.*

6 Applications

In this section we collect some applications of our main theorem. These include in particular the corresponding statements about (classical) Cartier modules on a scheme over a field of positive characteristic. We also construct a perverse t-structure on classical Cartier modules. As the perverse t-structure is usually denoted by cohomological notation in the literature, we chose to do the same here although we used homological notation before.

Notation 6.1. Let X be a scheme. Let $\mathrm{QCoh}(X)$ be the category of quasi-coherent $\mathcal{O}_X$-modules. We denote by $D(X)$ the (ordinary) derived category of $\mathrm{QCoh}(X)$. The full subcategory of $D(X)$ consisting of those objects that have coherent cohomology is denoted by $D_{\mathrm{coh}}(X)$. The full subcategory of $D(X)$ consisting of those objects that have bounded cohomology is denoted by

$$D^b(X) \coloneqq \bigcup_{\substack{a\to\infty \\ b\to-\infty}} D^{\leq a}(X) \cap D^{\geq b}(X)\,.$$

Similarly, denote by

$$D^+(X) \coloneqq \bigcup_{b\to-\infty} D^{\geq b}(X)$$

the full subcategory of those objects that have bounded below cohomology.

If $p > 0$ and X is an $\mathbb{F}_p$-scheme then we denote by $F\colon X \to X$ the absolute Frobenius of X. Consider the pushforward functor $F_*\colon\ \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$. We denote by $\mathrm{Cart}(X) \coloneqq \mathrm{Cart}(\mathrm{QCoh}(X), F_*)$ the category of (classical) Cartier modules and by $\mathrm{Frob}(X) \coloneqq \mathrm{Frob}(\mathrm{QCoh}(X), F_*)$ the category of (classical) Frobenius modules.

The functor F_* is exact since F is an affine morphism. Hence, the functor F_* induces a functor $F_*\colon D^b(X) \to D^b(X)$ which is t-exact with respect to the standard t-structures. If F is moreover finite, F_* restricts to a t-exact functor $F_*\colon D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(X)$.

Corollary 6.2. *Let $p > 0$ and X be an $\mathbb{F}_p$-scheme. Then there is a canonical equivalence of ∞-categories $\mathcal{D}(\mathrm{Cart}(X)) \xrightarrow{\simeq} \mathrm{Cart}(\mathcal{D}(X), \mathcal{D}(F_*))$.*

Proof. Apply Theorem 5.1 with the Grothendieck abelian category $\mathrm{QCoh}(X)$ and the exact and colimit-preserving functor $F_*\colon\ \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$. $\square$

Corollary 6.3. *Let $p > 0$ and X be a regular Noetherian $\mathbb{F}_p$-scheme. Then there is a canonical equivalence of ∞-categories $\mathcal{D}(\mathrm{Frob}(X)) \xrightarrow{\simeq} \mathrm{Frob}(\mathcal{D}(X), \mathcal{D}(F_*))$.*

Proof. First note that the functor $F^*\colon X \to X$ is left adjoint to F_* and exact by [Kun69, Theorem 2.1]. Thus we can apply Corollary 5.6 to $\mathcal{A} = \mathrm{QCoh}(X)$ and $G = F_*$. $\square$

Another application of Theorem 5.1 is that it enables us to define t-structures on $\mathcal{D}(\mathrm{Cart}(\mathcal{A}, F))$ by giving an induced t-structure on $\mathrm{Cart}(\mathcal{D}(\mathcal{A}), \mathcal{D}(F))$ in the sense of Section 3.

As an example we construct a perverse t-structure on the bounded derived category of classical Cartier modules with coherent cohomology. Perverse t-structures in general were first studied in [BBD82] and later more generally by Gabber [Gab04]. As they are usually denoted by cohomological notation in the literature, we chose to do the same here although we used homological notation before.

We first introduce some notation. For a scheme X and a point $x \in X$ we denote by $i_x\colon \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ the inclusion of the local ring. Recall that there is an exact inverse image functor

$$i_x^*\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(\mathrm{Spec}(\mathcal{O}_{X,x}))$$

and a left exact functor

$$i_x^!\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(\mathrm{Spec}(\mathcal{O}_{X,x}))$$

given by taking sections with support in $\overline{\{x\}}$ (compare with [Har66, Section IV.1, Variation 8]). We denote their derived, resp. right derived functors by

$$i_x^*\colon D(X) \to D(\mathrm{Spec}(\mathcal{O}_{X,x}))\,,$$

resp.

$$Ri_x^!\colon D^+(X) \to D^+(\mathrm{Spec}(\mathcal{O}_{X,x}))\,.$$

Theorem 6.4 ([Gab04, Theorem 9.1], [AB21, Theorem 3.10])**.** *Let X be a Noetherian scheme that admits a dualizing complex (cf. [Sta24, Tag 0A85]). Let $p\colon X \to \mathbb{Z}$ be a monotone and comonotone perversity function in the sense of [AB21, Definition 3.9]. Define full subcategories ${}^pD^{\geq 0}(X)$ and ${}^pD^{\leq 0}(X)$ of $D^b_{\mathrm{coh}}(X)$ as follows:*

An object $M \in D^b_{\mathrm{coh}}(X)$ is an object of ${}^pD^{\leq 0}(X)$ if for every point $x \in X$ we have $i_x^*(M) \in D^{\leq p(x)}(\mathrm{Spec}(\mathcal{O}_{X,x}))$.

An object $M \in D^b_{\mathrm{coh}}(X)$ is an object of ${}^pD^{\geq 0}(X)$ if for every point $x \in X$ we have $Ri_x^!(M) \in D^{\geq p(x)}(\mathrm{Spec}(\mathcal{O}_{X,x}))$.

These subcategories define a t-structure on $D^b_{\mathrm{coh}}(X)$, which is called the perverse t-structure *(with respect to p).*

Observe that the standard t-structure is a special case of a perverse t-structure.

Lemma 6.5. *Let X be a Noetherian scheme that admits a dualizing complex. The perverse t-structure with respect to the zero function is equivalent to the standard t-structure on $D^b_{\mathrm{coh}}(X)$.*

Proof. Let $M \in D^b_{\mathrm{coh}}(X)$ and $x \in X$. As t-structures are uniquely determined by their connective part it suffices to show ${}^0D^{\leq 0}(X) = D^{\leq 0}(X)$. This can be checked locally on X where $i_x^* M$ is given by $M \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}$. In particular, we have $H^j(i_x^* M) \cong i_x^* H^j(M)$ as localizations are flat.

Using this we can immediately conclude $D^{\leq 0}(X) = {}^0D^{\leq 0}(X)$ since the functors i_x^* are jointly conservative. □

Recall from [Sta24, Tag 0AU3] that for a Noetherian scheme X that admits a dualizing complex $\omega_X^\bullet$ there is a duality functor

$$\mathbb{D} := R\underline{Hom}_X(-, \omega_X^\bullet) \colon D^b_{\mathrm{coh}}(X)^{\mathrm{op}} \to D^b_{\mathrm{coh}}(X)\,.$$

It is an equivalence and its own inverse. In particular, we get a t-structure on $D^b_{\mathrm{coh}}(X)$ by applying $\mathbb{D}$ to the standard t-structure.

Moreover, note that by [Har66, Section V.7] the complex $Ri_x^! \omega_X^\bullet$ is concentrated in a single homological degree $d(x)$ for each point $x \in X$. The obtained map $d \colon X \to \mathbb{Z}$ is a monotone and comonotone perversity function. In the following, each perverse t-structure is the one associated to the function d.

Proposition 6.6. *Let X be a Noetherian scheme that admits a dualizing complex. The two t-structures $({}^dD^{\leq 0}(X), {}^dD^{\geq 0}(X))$ and $(\mathbb{D}(D^{\geq 0}(X)), \mathbb{D}(D^{\leq 0}(X)))$ on $D^b_{\mathrm{coh}}(X)$ coincide.*

Proof. It suffices to check that the coconnective parts of the two t-structures agree. By [AB21, Lemma 3.3(a)] we have $\mathbb{D}({}^0D^{\leq 0}(X)) = {}^dD^{\geq 0}(X)$. Thus, the proposition follows from Lemma 6.5. □

Using the description of the perverse t-structure with the dualizing complex, we can now define the induced perverse t-structure on classical Cartier modules. To be able to apply the results from Section 3 we first have to show that the Frobenius pushforward functor is right t-exact.

Lemma 6.7. *Let $p > 0$ and X be a Noetherian $\mathbb{F}_p$-scheme that admits a dualizing complex. Assume that the absolute Frobenius F of X is finite. Then $F_* \colon D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(X)$ is t-exact with respect to the perverse t-structure with respect to the function d described above.*

Proof. Note that the functor $F_* \colon D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(X)$ is t-exact with respect to the standard t-structure. Using this, the lemma follows directly from Proposition 6.6 and the fact that F_* commutes with the duality functor $\mathbb{D}$ by [Sta24, Tag 0AU3]. □

Definition 6.8. Let $p > 0$ and X be an $\mathbb{F}_p$-scheme. A (classical) Cartier module $M \in \mathrm{Cart}(X)$ is called *coherent* if its underlying $\mathcal{O}_X$-module UM is coherent.

We denote by $\mathcal{D}_{\mathrm{coh}}(\mathrm{Cart}(X))$ the full subcategory of $\mathcal{D}(\mathrm{Cart}(X))$ on those objects that have coherent cohomology.

Theorem 6.9. *Let $p > 0$ and X be a Noetherian $\mathbb{F}_p$-scheme that admits a dualizing complex. Assume that the absolute Frobenius F of X is finite.*

Then the perverse t-structure with respect to the function d described above induces a perverse t-structure on $\mathcal{D}^b_{\mathrm{coh}}(\mathrm{Cart}(X))$ such that the forgetful functor $\mathcal{D}(U)\colon \mathcal{D}^b_{\mathrm{coh}}(\mathrm{Cart}(X)) \to \mathcal{D}^b_{\mathrm{coh}}(X)$ is t-exact for the perverse t-structures.

This t-structure coincides with the t-structure of [Bau23, Definition 5.2.1].

Proof. We showed in Theorem 5.1 that there is an equivalence of ∞-categories

$$\alpha\colon \mathcal{D}(\mathrm{Cart}(X)) \to \mathrm{Cart}(\mathcal{D}(X), F_*)$$

which is t-exact with respect to the standard t-structure on $\mathcal{D}(\mathrm{Cart}(X))$ and the induced t-structure on $\mathrm{Cart}(\mathcal{D}(X), F_*)$. Because of the t-exactness α restricts to an equivalence

$$\mathcal{D}^b(\mathrm{Cart}(X)) \xrightarrow{\sim} \mathrm{Cart}(\mathcal{D}(X), F_*)^b$$

between the bounded objects of both ∞-categories. By the definition of the induced t-structure it follows that $\mathrm{Cart}(\mathcal{D}(X), F_*)^b$ is equivalent to the ∞-category $\mathrm{Cart}(\mathcal{D}^b(X), F_*)$.

Moreover, the coherence condition for the cohomology can be checked after applying the forgetful functor since the latter is exact. So, by the definition of α and the preceding discussion, we get an equivalence

$$\mathcal{D}^b_{\mathrm{coh}}(\mathrm{Cart}(X)) \xrightarrow{\sim} \mathrm{Cart}(\mathcal{D}^b_{\mathrm{coh}}(X), F_*)$$

of ∞-categories. Using this, it is left to show that there is an induced perverse t-structure on $\mathrm{Cart}(\mathcal{D}^b_{\mathrm{coh}}(X), F_*)$. This follows directly from Lemma 6.7 and Proposition 3.3.

The t-exactness of the forgetful functor $\mathcal{D}(U)\colon \mathcal{D}^b_{\mathrm{coh}}(\mathrm{Cart}(X)) \to \mathcal{D}^b_{\mathrm{coh}}(X)$ follows from the definition of α because $\mathcal{D}(U)$ can be written as the composition

$$\mathcal{D}^b_{\mathrm{coh}}(\mathrm{Cart}(X)) \xrightarrow{\alpha} \mathrm{Cart}(\mathcal{D}^b_{\mathrm{coh}}(X), F_*) \xrightarrow{U} \mathcal{D}^b_{\mathrm{coh}}(X)$$

of two t-exact functors.

That this t-structure coincides with the one of [Bau23, Definition 5.2.1] is clear from the construction. □

A Derived ∞-categories

In this appendix we state and prove some facts about derived ∞-categories of Grothendieck abelian categories that we use to prove our main theorem.

Recall that a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ on an ∞-category $\mathcal{C}$ is called right complete if the natural functor

$$\mathcal{C} \xrightarrow{\simeq} \lim(\cdots \to \mathcal{C}_{\geq 0} \xrightarrow{\tau_{\geq 1}} \mathcal{C}_{\geq 1} \xrightarrow{\tau_{\geq 2}} \mathcal{C}_{\geq 2} \to \dots)$$

is an equivalence of ∞-categories. It is called right separated if $\bigcap_n \mathcal{C}_{\leq n} = 0$ and left separated if $\bigcap_n \mathcal{C}_{\geq n} = 0$.

Notation A.1. Let $\mathcal{A}$ be a Grothendieck abelian category. We denote by $\mathcal{D}(\mathcal{A})$ the derived ∞-category of $\mathcal{A}$, cf. [Lur17a, Definition 1.3.5.8].

It carries a t-structure $(\mathcal{D}(\mathcal{A})_{\geq 0}, \mathcal{D}(\mathcal{A})_{\leq 0})$ (cf. [Lur17a, Definition 1.3.5.16]) which is accessible, right complete and compatible with filtered colimits by [Lur17a, Proposition 1.3.5.21].

For each n we denote by $\iota_{\geq n} \colon \mathcal{D}(\mathcal{A})_{\geq n} \leftrightarrows \mathcal{D}(\mathcal{A}) \colon \tau_{\geq n}$ the adjunction of the inclusion and the connective cover functor and by $\tau_{\leq n} \colon \mathcal{D}(\mathcal{A}) \leftrightarrows \mathcal{D}(\mathcal{A})_{\leq n} \colon \iota_{\leq n}$ the adjunction of the truncation functor and the inclusion.

Furthermore, the heart of the t-structure is given by $\mathcal{D}(\mathcal{A})^{\heartsuit} \simeq \mathcal{A}$ (cf. [Lur18, Remark C.5.4.11]) and we denote the inclusion of the heart by $H \colon \mathcal{A} \to \mathcal{D}(\mathcal{A})$.

The next proposition can be seen as a universal property of derived ∞-categories. It shows that t-exact and colimit-preserving functors out of a derived ∞-category are uniquely determined by their restrictions to the heart.

Proposition A.2. *Let $\mathcal{A}$ be a Grothendieck abelian category and $\mathcal{C}$ a presentable stable ∞-category. Assume that $\mathcal{C}$ carries a right complete and left separated t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ that is compatible with filtered colimits. Then the restriction functor*

$$\mathrm{LFun}^{\text{t-ex}}(\mathcal{D}(\mathcal{A}), \mathcal{C}) \to \mathrm{LFun}^{\mathrm{ex}}(\mathcal{A}, \mathcal{C}^{\heartsuit}), \quad G \mapsto \pi_0 GH$$

is an equivalence, where objects in the source are colimit-preserving t-exact functors from $\mathcal{D}(\mathcal{A})$ to $\mathcal{C}$ and objects in the target are colimit-preserving exact functors from $\mathcal{A}$ to $\mathcal{C}^{\heartsuit}$.

Proof. By [Lur18, Theorem C.5.4.9] the restriction to the heart induces an equivalence

$$\mathrm{LFun}^{\mathrm{lex}}(\mathcal{D}(\mathcal{A})_{\geq 0}, \mathcal{C}_{\geq 0}) \xrightarrow{\simeq} \mathrm{LFun}^{\mathrm{ex}}(\mathcal{A}, \mathcal{C}^{\heartsuit})$$

where objects in the source are colimit-preserving and left exact functors from $\mathcal{D}(\mathcal{A})_{\geq 0}$ to $\mathcal{C}_{\geq 0}$. Therefore, it suffices to show that the restriction functor

$$\mathrm{LFun}^{\text{t-ex}}(\mathcal{D}(\mathcal{A}), \mathcal{C}) \to \mathrm{LFun}^{\mathrm{lex}}(\mathcal{D}(\mathcal{A})_{\geq 0}, \mathcal{C}_{\geq 0})$$

is an equivalence.

By [Lur17a, Proposition 1.3.5.21(3)] the t-structure on $\mathcal{D}(\mathcal{A})$ is right complete, i.e. $\mathcal{D}(\mathcal{A}) \xrightarrow{\simeq} \lim_n \mathcal{D}(\mathcal{A})_{\geq -n}$. Thus, combining [Lur18, Remark C.1.2.10] and [Lur18, Proposition C.3.1.1] we get an equivalence

$$\mathrm{LFun}^{\text{r-t-ex}}(\mathcal{D}(\mathcal{A}), \mathcal{C}) \xrightarrow{\simeq} \mathrm{LFun}(\mathcal{D}(\mathcal{A})_{\geq 0}, \mathcal{C}_{\geq 0})$$

where objects in the source are colimit-preserving right t-exact functors from $\mathcal{D}(\mathcal{A})$ to $\mathcal{C}$ and objects in the target are colimit-preserving functors from $\mathcal{D}(\mathcal{A})_{\geq 0}$ to $\mathcal{C}_{\geq 0}$. But by [Lur18, Proposition C.3.2.1] this induces an equivalence

$$\mathrm{LFun}^{\text{t-ex}}(\mathcal{D}(\mathcal{A}), \mathcal{C}) \xrightarrow{\simeq} \mathrm{LFun}^{\mathrm{lex}}(\mathcal{D}(\mathcal{A})_{\geq 0}, \mathcal{C}_{\geq 0})\,.$$

□

In particular, by [Lur18, Remark C.5.4.11] the above lemma applies if we take $\mathcal{C}$ to be the derived ∞-category $\mathcal{D}(\mathcal{B})$ of a Grothendieck abelian category $\mathcal{B}$.

Notation A.3. Let $\mathcal{A}$ and $\mathcal{C}$ be as in Proposition A.2. We denote by

$$\mathcal{D}\colon \operatorname{LFun}^{\mathrm{ex}}(\mathcal{A},\mathcal{C}^{\heartsuit}) \to \operatorname{LFun}^{\text{t-ex}}(\mathcal{D}(\mathcal{A}),\mathcal{C}), \quad G \mapsto \mathcal{D}(G)$$

the inverse to the restriction functor.

In the following we investigate the properties of the functor $\mathcal{D}(G)$ if $G\colon \mathcal{A} \to \mathcal{C}^{\heartsuit}$ is an exact and colimit-preserving functor. In the case where $\mathcal{C}$ is also a derived ∞-category, we show that $\mathcal{D}$ preserves adjoint functors and monadicity. To that end we start by showing that $\mathcal{D}$ preserves conservativity of functors. For that we need the following lemma.

Lemma A.4. *Let $\mathcal{A}$ and $\mathcal{C}$ be as in Proposition A.2 and $G\colon \mathcal{A} \to \mathcal{C}^{\heartsuit}$ an exact and colimit-preserving functor. Then there is an equivalence $\pi_n\mathcal{D}(G) \simeq G\pi_n$ of functors $\mathcal{D}(\mathcal{A}) \to \mathcal{C}^{\heartsuit}$ for each n.*

Proof. We show the claim for $n = 0$. Then the lemma follows from

$$\pi_n\mathcal{D}(G) \simeq \pi_0\Sigma^{-n}\mathcal{D}(G) \simeq \pi_0\mathcal{D}(G)\Sigma^{-n} \simeq G\pi_0\Sigma^{-n} \simeq G\pi_n$$

where we used that $\mathcal{D}(G)$ is exact in the second equivalence.
Note that by the definition of the restriction functor of Proposition A.2 we have $G \simeq \pi_0\mathcal{D}(G)H$. We show that there are equivalences

$$\pi_0\mathcal{D}(G)H\pi_0 \xrightarrow{\simeq} \pi_0\mathcal{D}(G)\iota_{\leq 0}\tau_{\leq 0} \xleftarrow{\simeq} \pi_0\mathcal{D}(G) \tag{5}$$

where the first map is given by the counit of the adjunction $\iota_{\geq 0} \dashv \tau_{\geq 0}$ (note that $H\pi_0 \simeq \iota_{\geq 0}\tau_{\geq 0}\iota_{\leq 0}\tau_{\leq 0}$) and the second map is given by the unit of the adjunction $\tau_{\leq 0} \dashv \iota_{\leq 0}$.

To see that the first map is an equivalence consider the fiber sequence

$$H\pi_0X = \iota_{\geq 0}\tau_{\geq 0}X \to X \to \iota_{\leq -1}\tau_{\leq -1}X$$

for $X \in \mathcal{D}(\mathcal{A})_{\leq 0}$. Applying $\mathcal{D}(G)$ we get another fiber sequence

$$\mathcal{D}(G)H\pi_0X \to \mathcal{D}(G)X \to \mathcal{D}(G)\iota_{\leq -1}\tau_{\leq -1}X$$

where the last term is (-1)-truncated and the other terms are 0-truncated because of the t-exactness of $\mathcal{D}(G)$. This shows that the induced map

$$\pi_0\mathcal{D}(G)H\pi_0X \to \pi_0\mathcal{D}(G)X$$

is an equivalence. Hence, also the first map of (5) is an equivalence.

To check that the second map of (5) is an equivalence we consider the fiber sequence

$$\iota_{\geq 1}\tau_{\geq 1}Y \to Y \to \iota_{\leq 0}\tau_{\leq 0}Y$$

for $Y \in \mathcal{D}(\mathcal{A})$. Applying $\mathcal{D}(G)$ we get another fiber sequence whose first term is 1-connective and whose last term is 0-truncated. Hence, the unit of the adjunction $\tau_{\leq 0} \dashv \iota_{\leq 0}$ induces an equivalence $\pi_0\mathcal{D}(G)Y \simeq \pi_0\mathcal{D}(G)\tau_{\leq 0}Y$. So, also the second map of (5) is an equivalence, which completes the proof. □

Using the above lemma we can conclude that $\mathcal{D}$ preserves conservativity of functors. Note that if an ∞-category $\mathcal{C}$ carries a right and left separated t-structure then a morphism f in $\mathcal{C}$ is an equivalence if and only if for every n, the induced map $\pi_n(f)$ is an equivalence.

Corollary A.5. *Let $\mathcal{A}$ and $\mathcal{C}$ be as in Proposition A.2 and $G\colon \mathcal{A} \to \mathcal{C}^\heartsuit$ an exact and colimit-preserving functor that is conservative. Then the induced functor $\mathcal{D}(G)$ is conservative.*

Proof. Let f be a morphism in $\mathcal{D}(\mathcal{A})$ such that $\mathcal{D}(G)(f)$ is an equivalence. It follows that for each $n \in \mathbb{Z}$ the morphism $\pi_n\mathcal{D}(G)(f)$ is an equivalence. But by Lemma A.4 this morphism is equivalent to the morphism $G\pi_n(f)$. Using that G is conservative we get that $\pi_n(f)$ is an equivalence for every n. Note that the t-structure on $\mathcal{D}(\mathcal{A})$ is right complete by [Lur17a, Proposition 1.3.5.21(3)], so in particular right separated by the dual of [Lur17a, Proposition 1.2.1.19]. Furthermore, it is left separated by [Lur18, Remark C.5.4.11]. Thus, $\pi_n(f)$ being an equivalence already implies that f is an equivalence. □

We finish this section by showing that $\mathcal{D}$ preserves adjoints and monadicity.

Lemma A.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be Grothendieck abelian categories and $L\colon \mathcal{A} \to \mathcal{B}$ and $R\colon \mathcal{B} \to \mathcal{A}$ be exact and colimit-preserving functors such that L is left adjoint to R. Then $\mathcal{D}(L)$ is left adjoint to $\mathcal{D}(R)$.*

Proof. We define the unit and counit as

$$u_{\mathcal{D}}\colon \mathrm{id}_{\mathcal{D}(\mathcal{A})} \simeq \mathcal{D}(\mathrm{id}_{\mathcal{A}}) \xrightarrow{\mathcal{D}(u)} \mathcal{D}(RL) \simeq \mathcal{D}(R)\mathcal{D}(L)$$

and

$$c_{\mathcal{D}}\colon \mathcal{D}(L)\mathcal{D}(R) \simeq \mathcal{D}(LR) \xrightarrow{\mathcal{D}(c)} \mathcal{D}(\mathrm{id}_{\mathcal{B}}) \simeq \mathrm{id}_{\mathcal{D}(\mathcal{B})}\,,$$

respectively, where $u\colon \mathrm{id}_{\mathcal{A}} \to RL$ and $c\colon LR \to \mathrm{id}_{\mathcal{B}}$ denote the unit and counit of the adjunction $L \dashv R$. To check that they satisfy the triangle identities (which is enough by [RV15, Remark 4.4.5]) consider the following diagram whose commutativity can be checked on the heart by using the equivalence from Proposition A.2

$$\begin{array}{ccccccc}
\mathcal{D}(L) & \xrightarrow{\quad u_{\mathcal{D}}\quad} & & \mathcal{D}(L)\mathcal{D}(R)\mathcal{D}(L) & & \xrightarrow{\quad c_{\mathcal{D}}\quad} & \mathcal{D}(L) \\
\downarrow{\simeq} & & \swarrow{\simeq} & & \searrow{\simeq} & & \downarrow{\simeq} \\
\mathcal{D}(L)\mathcal{D}(\mathrm{id}_{\mathcal{A}}) & \xrightarrow{\mathcal{D}(u)} & \mathcal{D}(L)\mathcal{D}(RL) & & \mathcal{D}(LR)\mathcal{D}(L) & \xrightarrow{\mathcal{D}(c)} & \mathcal{D}(\mathrm{id}_{\mathcal{B}})\mathcal{D}(L) \\
\downarrow{\simeq} & & \searrow{\simeq} & & \swarrow{\simeq} & & \downarrow{\simeq} \\
\mathcal{D}(L\,\mathrm{id}_{\mathcal{A}}) & \xrightarrow[\mathcal{D}(Lu)]{\qquad} & & \mathcal{D}(LRL) & & \xrightarrow[\mathcal{D}(cL)]{\qquad} & \mathcal{D}(\mathrm{id}_{\mathcal{B}}\,L).
\end{array}$$

Note that the lower horizontal composition is equivalent to $\mathcal{D}(cL \circ Lu)$ which is equivalent to the identity by a triangle identity for u and c. Thus, the top horizontal composition in the diagram is equivalent to the identity. The other triangle identity can be shown analogously. □

Remark A.7. *One can also extend the functor $\mathcal{D}$ to a functor from the $(2,2)$-category of Grothendieck abelian categories with colimit-preserving exact functors to the $(\infty,2)$-category of stable ∞-categories with t-structure with colimit-preserving t-exact functors. From this description the result of Lemma A.6 is a formal consequence.*

Corollary A.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be Grothendieck abelian categories and $L\colon \mathcal{A} \to \mathcal{B}$ and $R\colon \mathcal{B} \to \mathcal{A}$ be exact and colimit-preserving functors such that L is left adjoint to R. Assume that R exhibits $\mathcal{B}$ as monadic over $\mathcal{A}$. Then $\mathcal{D}(R)$ exhibits $\mathcal{D}(\mathcal{B})$ as monadic over $\mathcal{D}(\mathcal{A})$.*

Proof. Note that by Lemma A.6 the functor $\mathcal{D}(R)$ has a left adjoint. By the ∞-categorical Barr-Beck Theorem (cf. [Lur17a, Theorem 4.7.3.5]) it is enough to show that $\mathcal{D}(R)$ is conservative, $\mathcal{D}(\mathcal{B})$ admits colimits (of $\mathcal{D}(R)$-split simplicial objects) and that $\mathcal{D}(R)$ preserves these colimits. Using the ∞-categorical Barr-Beck Theorem for R we get that R is conservative. Hence, $\mathcal{D}(R)$ is conservative by Corollary A.5. Moreover, the ∞-category $\mathcal{D}(\mathcal{B})$ admits all colimits by [Lur17a, Proposition 1.3.5.21(1)], and $\mathcal{D}(R)$ preserves all colimits by definition. □

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