

Unstable étale motives

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Abstract

We prove a rigidity result for certain p -complete étale \mathbb{A}^1 -invariant sheaves of anima over a qcqs finite-dimensional base scheme S of bounded étale cohomological dimension with p invertible on S . This generalizes results of Suslin–Voevodsky [SV96], Ayoub [Ayo14], Cisinski–Déglise [CD15], and Bachmann [Bac21a, BH21] to the unstable setting. Over a perfect field we exhibit a large class of sheaves to which our main theorem applies, in particular the p -completion of the étale sheafification of any 2-effective 2-connective motivic space, as well as the p -completion of any 4-connective \mathbb{A}^1 -invariant étale sheaf. We use this rigidity result to prove (a weaker version of) an étale analog of Morel’s theorem stating that for a Nisnevich sheaf of abelian groups, strong \mathbb{A}^1 -invariance implies strict \mathbb{A}^1 -invariance. Moreover, this allows us to construct an unstable étale realization functor on 2-effective 2-connective motivic spaces.

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1 Introduction

It is well-known that étale cohomology with locally constant \mathbb{F}_p -coefficients is \mathbb{A}^1 -invariant, cf. [GAV71, Corollaire XV.2.2]. One can ask whether the converse is true, resulting in rigidity results of Suslin–Voevodsky [SV96], Ayoub [Ayo14] or Cisinski–Déglise [CD15]. The most general of such results is the following version for spectral coefficients, due to Bachmann.

Theorem ([Bac21a, BH21]). *Let S be a scheme, and p a prime invertible on S . There are canonical equivalences*

$$\mathrm{Shv}_{\mathrm{ét}}^h(S_{\mathrm{ét}}, \mathrm{Sp})_p^\wedge \xrightarrow{\cong} \mathcal{SH}_{\mathrm{ét}}^{S^1}(S)_p^\wedge \xrightarrow{\cong} \mathcal{SH}_{\mathrm{ét}}(S)_p^\wedge$$

between p -complete hypersheaves of spectra on the small étale site of S , p -complete \mathbb{A}^1 -invariant étale hypersheaves of spectra on Sm_S , and its (p -complete) \mathbb{G}_m -stabilization.

The main goal of this article is a generalization of Bachmann’s result to the unstable setting. We write $\mathrm{cd}(k)$ for the étale cohomological dimension of a field k . Our main theorem is the following.

Theorem A (Corollary 8.8). *Let k be a perfect field with $\mathrm{cd}(k) < \infty$, and $p \neq \mathrm{char}(k)$ be a prime. There is a canonical equivalence*

$$L_p\iota! : \left(\mathrm{Shv}_{\mathrm{ét}}^h(k_{\mathrm{ét}})_{*, \geq 4} \right)_p^\wedge \xrightarrow{\cong} \left(\mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_k)_{*, \mathbb{A}^1, \geq 4} \right)_p^\wedge.$$

Here, $(\mathrm{Shv}_{\mathrm{ét}}^h(k_{\mathrm{ét}})_{*, \geq 4})_p^\wedge$ denotes the full subcategory of $\mathrm{Shv}_{\mathrm{ét}}^h(k_{\mathrm{ét}})_*$ (the ∞ -topos of étale hypersheaves on $k_{\mathrm{ét}}$) consisting of those sheaves that are the p -completion of a 4-connective sheaf, and similarly $(\mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_k)_{*, \mathbb{A}^1, \geq 4})_p^\wedge$ denotes the full subcategory of $\mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_k)_*$ consisting of those sheaves that are the p -completion of an \mathbb{A}^1 -invariant 4-connective sheaf.

Remark 1.1. The above result is likely true for integers $2 \leq m < 4$, but our methods are not strong enough to show this. Note however that the functor

$$L_p\iota! : \left(\mathrm{Shv}_{\mathrm{ét}}^h(k_{\mathrm{ét}})_* \right)_p^\wedge \rightarrow \left(\mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_k)_{*, \mathbb{A}^1} \right)_p^\wedge$$

is *not* an equivalence. Indeed, any sheaf of sets is automatically p -complete (cf. [Mat24b, Lemma 3.13]), but there are \mathbb{A}^1 -invariant étale sheaves of sets that are not coming from the small étale site, e.g. \mathbb{G}_m .

We will deduce this theorem from a related version that holds over any base scheme with some bound on the étale cohomological dimension of its residue fields. To state the result, we need the following definition:

Definition (Definition 3.10). Let S be a scheme, and $X \in \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S)$. We say that X is p -completely étale \mathbb{A}^1 -nilpotent if there exists a highly connected tower $(X_n)_n$ under X (i.e. the connectivity of the transition maps $X_{n+1} \rightarrow X_n$ goes to infinity as $n \rightarrow \infty$) such that the following holds:

1. $X_0 \cong *$.
2. $X \cong \lim_n X_n$.

3. Each of the morphisms $X_{n+1} \rightarrow X_n$ is part of a fiber sequence $X_{n+1} \rightarrow X_n \rightarrow K_n$ where $K_n \cong \Omega_*^\infty E_n$ is a connected infinite loop sheaf for some $E_n \in \mathrm{Sp}(\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S))_{\geq 1}$.
4. For every n the object $L_p E_n = \lim_k E_n // p^k$ is \mathbb{A}^1 -invariant.

In other words, an sheaf X is p -completely étale \mathbb{A}^1 -nilpotent if it admits a principal refinement of (a version of) the Postnikov-tower of X , such that the layers are \mathbb{A}^1 -invariant after p -completion.

Theorem B (Corollary 6.7 and Lemma 3.13). *Let S be a qcqs scheme of finite Krull-dimension with $\sup_{s \in S} \mathrm{cd}(s) < \infty$, and let p be a prime invertible on S . The functor*

$$\iota_!^p : \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(S_{\mathrm{\acute{e}t}})_{p, \mathrm{nil}}^\wedge \rightarrow \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)_p^\wedge$$

is fully faithful, with essential image exactly the p -completions of p -completely étale \mathbb{A}^1 -nilpotent sheaves. In particular, every sheaf in the essential image is \mathbb{A}^1 -invariant.

We will deduce Theorem A from Theorem B by doing a careful analysis of which étale sheaves are actually p -completely étale \mathbb{A}^1 -nilpotent, which we will now explain.

Étale \mathbb{A}^1 -nilpotent sheaves over a perfect field

Over a perfect field k , by a result of Asok–Fasel–Hopkins, in the Nisnevich local world there is no difference between Nisnevich nilpotence and Nisnevich \mathbb{A}^1 -nilpotence:

Lemma ([AFH22, Proposition 3.2.3] and [Mat24b, Lemma 5.19]). *Let k be a perfect field and $X \in \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_k)_*$ be connected and \mathbb{A}^1 -invariant. Then X is nilpotent (in the topos theoretic sense, i.e. there exists a principal refinement of the Postnikov tower of X) if and only if X is \mathbb{A}^1 -nilpotent (i.e. there exists a principal refinement of the Postnikov tower of X with \mathbb{A}^1 -invariant layers).*

This naturally leads to the following analogous question:

Question 1.2. *Let k be a perfect field, $p \neq \mathrm{char}(k)$ be a prime, and $X \in \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_k)_*$ be connected \mathbb{A}^1 -invariant. If X is étale nilpotent, is it true that X is $(p$ -completely) étale \mathbb{A}^1 -nilpotent?*

We are able to give a partial answer to this question in form of the following result, which is interesting on its own:

Theorem C (Proposition 8.4). *Let k be a perfect field of exponential characteristic e with $\mathrm{cd}(k) < \infty$, $p \neq e$ a prime, and $X \in \mathrm{Spc}(k)_*$ be a motivic space that is 2-effective, nilpotent (as a Nisnevich sheaf) and $\mathbb{Z}[\frac{1}{e}]$ -local. Then $L_{\mathrm{\acute{e}t}} X$ is étale \mathbb{A}^1 -nilpotent (so in particular \mathbb{A}^1 -invariant and p -completely étale \mathbb{A}^1 -nilpotent).*

Note that this is a priori surprising, as, in general, the étale sheafification of an \mathbb{A}^1 -invariant presheaf is no longer \mathbb{A}^1 -invariant, and simultaneous étale sheafification and \mathbb{A}^1 -localization is computed by the countable colimit of the alternating application of the two localization functors. The above theorem

says that if we start with a nilpotent and 2-effective motivic space, then this procedure is unnecessary, as the étale sheafification is already \mathbb{A}^1 -invariant. The proof makes heavy use of the theory of \mathbb{P}^1 -Postnikov towers as developed in [ABH23].

Using this, we can give more examples of étale sheaves for which rigidity holds. Bachmann showed that in p -complete étale stable motivic homotopy theory over an algebraically closed field, there is an equivalence $L_p \Sigma^\infty \mathbb{G}_m \cong L_p \mathbb{S}^1$, cf. [Bac21a, Theorem 6.5] and [BH21, proof of Theorem 3.1] (note that if the field is algebraically closed, the twisting spectrum is trivialized, yielding the above claim). A similar statement is true unstably:

Theorem D (Theorems 7.8 and 7.10). *Let k be an algebraically closed field with $p \neq \text{char}(k)$. Then there is a retract diagram (in $\text{Shv}_{\text{ét}}^h(\text{Sm}_k)$)*

$$L_p S^2 \xrightarrow{\tau} L_p L_{\text{ét}, \mathbb{A}^1} \mathbb{P}^1 \xrightarrow{\sigma} L_p S^2.$$

Moreover, after (the p -completion of) a twofold suspension, τ becomes an equivalence, i.e. we have

$$L_p S^4 \cong L_p L_{\text{ét}, \mathbb{A}^1} \Sigma^2 \mathbb{P}^1.$$

Using this retract, we can prove that in fact every 4-connective \mathbb{A}^1 -invariant étale sheaf over an algebraically closed field is (p -completely) a retract of a 2-effective and 2-connective étale motivic space. Using étale descent, this result suggests the following corollary.

Corollary E (Proposition 8.7). *Let k be a perfect field with $\text{cd}(k) < \infty$ and $p \neq \text{char}(k)$ a prime, and $X \in \text{Shv}_{\text{ét}}^h(\text{Sm}_k)_*$ be 4-connective and \mathbb{A}^1 -invariant. Then X is p -completely small, i.e. $L_{p!} X \cong L_p X$.*

Étale motivic spaces at the characteristic

If k is a field of characteristic $p > 0$, then the Artin–Schreier sequence $\mathbb{F}_p \rightarrow \mathbb{A}^1 \xrightarrow{1-F} \mathbb{A}^1$ shows that the \mathbb{A}^1 -localization of the constant sheaf of abelian groups \mathbb{F}_p is 0. From this, one can for example deduce the vanishing of the whole category of motives with p -torsion coefficients, cf. [CD15, Proposition A.3.1]. The strongest such vanishing result is due to Bachmann and Hoyois:

Theorem ([BH21, Theorem A.1]). $\mathcal{SH}_{\text{ét}}(S)_p^\wedge \cong \mathcal{SH}_{\text{ét}}^{S^1}(S)_p^\wedge \cong 0$ for S any scheme over \mathbb{F}_p .

Unstably, we can show the following version:

Theorem F (Proposition 5.1). *Let S be a qcqs \mathbb{F}_p -scheme of finite Krull-dimension with $\sup_{s \in S} \text{cd}(s) < \infty$, and $X \in \text{Shv}_{\text{ét}}^h(\text{Sm}_S)$ be p -completely étale \mathbb{A}^1 -nilpotent. Then $L_p X \cong *$.*

Related is the following result:

Theorem G (Corollary 5.4). *Let k be a perfect field of characteristic $p > 0$ with $\text{cd}(k) < \infty$. Let $A \in \text{Shv}_{\text{ét}}(\text{Sm}_k, \mathcal{A}b)$ be a sheaf of abelian groups, and $k \geq 3$. If $L_p K(A, k)$ is \mathbb{A}^1 -invariant, then $L_p K(A, k) \cong *$.*

Remark 1.3. It is not true that unstably the whole p -complete category vanishes: Indeed, étale sheaves of sets are always p -complete by [Mat24b, Lemma 3.13], but there are \mathbb{A}^1 -invariant sheaves of sets that are nontrivial, e.g. \mathbb{G}_m .

Étale hyperdescent for rational motivic spaces

Let k be a perfect field. Recall from e.g. [CD19, §16] that there is a splitting $\mathcal{SH}(k)_{\mathbb{Q}} \cong \mathcal{SH}(k)_{\mathbb{Q}}^+ \times \mathcal{SH}(k)_{\mathbb{Q}}^-$. Every object in $\mathcal{SH}(k)_{\mathbb{Q}}^+$ satisfies étale hyperdescent, whereas if the cohomological dimension of k is finite, $\mathcal{SH}(k)_{\mathbb{Q}}^- = 0$. In particular, one obtains the following result:

Theorem ([CD19, §16]). *Let k be a perfect field with $\mathrm{cd}(k) < \infty$. Then any rational motivic spectrum $E \in \mathcal{SH}(k)_{\mathbb{Q}}$ satisfies étale hyperdescent.*

Using similar techniques as in the proof of Theorem C, we can prove an unstable version of this result.

Theorem H (Theorem 4.2). *Let k be a perfect field with $\mathrm{cd}(k) < \infty$. Any rational nilpotent and 2-effective motivic space $X \in \mathrm{Spc}(k)_*$ satisfies étale hyperdescent.*

Strict \mathbb{A}^1 -invariance for étale sheaves of abelian groups

Let k be a perfect field. Recall that a Nisnevich sheaf of abelian groups A is 1-strictly (resp. strictly) Nisnevich \mathbb{A}^1 -invariant if $K_{\mathrm{nis}}(A, 1) \in \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_k)$ is \mathbb{A}^1 -invariant (resp. $K_{\mathrm{nis}}(A, n) \in \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_k)$ is \mathbb{A}^1 -invariant for all $n \geq 0$). Morel showed in [Mor12, Theorem 4.46] that 1-strictly Nisnevich \mathbb{A}^1 -invariant sheaves of abelian groups are already strictly Nisnevich \mathbb{A}^1 -invariant. This naturally leads to the analogous question for étale sheaves. Let us say that an étale sheaf of abelian groups A on Sm_k is *m -strictly étale \mathbb{A}^1 -invariant* for $m \geq 1$ if $K(A, m) \in \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_k)$ is \mathbb{A}^1 -invariant, and that it is *strictly étale \mathbb{A}^1 -invariant* if $K(A, n)$ is \mathbb{A}^1 -invariant for all $n \geq 0$.

Question 1.4. *Let k be a perfect field with $\mathrm{cd}(k) < \infty$ and A be an étale sheaf of abelian groups on Sm_k . Suppose that A is 1-strictly étale \mathbb{A}^1 -invariant. Is it true that A is strictly étale \mathbb{A}^1 -invariant?*

Using the rigidity result, we can partially answer this question.

Theorem I (Theorem 9.7). *Let k be a perfect field with $\mathrm{cd}(k) < \infty$ and $A \in \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_k, \mathcal{A}b)$ be an étale sheaf of abelian groups. If A is 4-strictly étale \mathbb{A}^1 -invariant, then A is strictly étale \mathbb{A}^1 -invariant.*

In other words, this is a weak version of Morel's theorem in the étale world.

Remark 1.5. It is unknown to the author if for $m \in \{1, 2, 3\}$ we still have the implication that m -strictly étale \mathbb{A}^1 -invariant sheaves are already strictly étale \mathbb{A}^1 -invariant. This is closely related to whether Theorem A holds for $m < 4$.

Using the equivalence $H_{\mathrm{ét}}^i(U, A) \cong \pi_{n-i}(K(A, n)(U))$ for every smooth k -scheme U , integers $n \geq i \geq 0$ and étale sheaf of abelian groups A , we can reformulate the above theorem as the following.

Corollary J. *Let k be a perfect field with $\mathrm{cd}(k) < \infty$ and $A \in \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S, \mathcal{A}b)$ be an étale sheaf of abelian groups. If $H_{\mathrm{ét}}^i(-, A)$ is \mathbb{A}^1 -invariant for every $i \in \{0, \dots, 4\}$, then $H_{\mathrm{ét}}^i(-, A)$ is \mathbb{A}^1 -invariant for every $i \geq 0$.*

Étale realization functor

As another application of this theory, we obtain an unstable étale realization functor:

Theorem K. *Let k be a perfect field with $\mathrm{cd}(k) < \infty$, and $p \neq \mathrm{char}(k)$ a prime. There exists an étale realization functor*

$$Re_p: \mathrm{Spc}(k)_{*,\mathrm{nil},2\text{-eff}} \rightarrow \mathrm{Shv}_{\mathrm{ét}}^h(k_{\mathrm{ét}})_{*,p}^{\wedge}.$$

Here, the left-hand side is the ∞ -category of nilpotent and 2-effective pointed motivic spaces.

Proof. Write e for the exponential characteristic of k . Let $X \in \mathrm{Spc}(k)_{*,\mathrm{nil},2\text{-eff}}$. Then also $L_{\mathbb{Z}[\frac{1}{e}]}X$ is nilpotent and 2-effective: Indeed, nilpotence was shown in [AFH22, Proposition 4.3.8] (they only show that $L_{\mathbb{Z}[\frac{1}{e}]}X$ is weakly \mathbb{A}^1 -nilpotent, but under the assumption that X is nilpotent their proof in fact shows that $L_{\mathbb{Z}[\frac{1}{e}]}X$ is nilpotent). That it is 2-effective was shown in [ABH23, Proposition 4.3.4]. Hence, $Y := L_{\mathrm{ét}}L_{\mathbb{Z}[\frac{1}{e}]}X$ is étale \mathbb{A}^1 -nilpotent by Theorem C, so in particular p -completely étale \mathbb{A}^1 -nilpotent by Lemma 3.11. We can therefore define Re_pX to be the inverse of the equivalence $\iota_!^p$ from Theorem B, applied to L_pY . \square

Improving the bounds

Let k be a perfect field and $\epsilon \geq 0$ be an integer. Recall that $\mathcal{SH}^{S^1}(k)(\epsilon)$, the category of ϵ -effective motivic S^1 -spectra, and $\mathcal{SH}(k)(\epsilon)$, the category of ϵ -effective motivic spectra both possess a t-structure [BY20, §6.1]. In particular, by [BY20, Lemma 6.2], there is an induced functor on the hearts

$$\omega^\infty: \mathcal{SH}(k)(\epsilon)^\heartsuit \rightarrow \mathcal{SH}^{S^1}(k)(\epsilon)^\heartsuit.$$

By [BY20, Bac21b, Fel21], this functor is an equivalence for $\epsilon \geq 2$ (see also [ABH23, Theorem 2.2.30 (2)] for a combined proof of this fact). This equivalence lets us resolve any ϵ -effective motivic space by a tower, where the layers are \mathbb{P}^1 -infinite loop spaces, cf. Proposition 2.14. All the bounds presented in the above introduction are derived from the fact that this works for $\epsilon = 2$. If it turns out that also for $\epsilon = 1$ the above functor is an equivalence, one can improve the above results as follows:

- In Theorem A, one can replace 4-connective by 3-connective.
- In Theorem C, one can replace 2-effective by 1-effective.
- In Corollary E, one can replace 4-connective by 3-connective.
- In Theorem H, one can replace 2-effective by 1-effective.
- In Theorem I, one can replace 4-strictly by 3-strictly.
- In Corollary J, one can replace $\{0, \dots, 4\}$ by $\{0, \dots, 3\}$.
- In Theorem K, one can replace 2-effective by 1-effective.

Linear Outline

We begin in Section 2 with background material.

Section 2.1 establishes a criterion for when a geometric morphism of ∞ -topoi preserves limits along certain highly connected towers.

In Section 2.2, we strengthen some results on unstable p -completion from [Mat24b, Mat24a].

In Section 2.3, we collect key results on \mathbb{P}^1 -Postnikov towers from the work of Asok–Bachmann–Hopkins [ABH23], which serve as an important tool in our later arguments.

Section 2.4 collects key facts about hypercomplete étale ∞ -topoi.

Finally, in Section 2.5, we show that any n -connective pointed sheaf in an ∞ -topos \mathcal{X} has a canonical resolution by “free n -connective pointed sheaves”, i.e. sheaves of the form $S^n \wedge T$ for $T \in \mathcal{X}_*$.

Section 3 examines the connection between étale sheaves and \mathbb{A}^1 -invariant presheaves. In particular, we introduce the notion of (p -completely) étale \mathbb{A}^1 -nilpotent sheaves, which play a central role in the rigidity theorem.

Section 4 shows that, after rationalization, étale and Nisnevich motivic spaces become closely related, culminating in Theorem H.

Section 5 provides some vanishing results of p -complete étale motivic spaces, if p is the characteristic of the base scheme, proving Theorems F and G.

The unstable rigidity theorem is proven in Section 6, where we establish Theorem B.

In Section 7, we construct a retract $S^2 \rightarrow \mathbb{G}_m \rightarrow S^2$ in the p -complete étale unstable motivic setting, yielding Theorem D.

Section 8 presents concrete examples of étale \mathbb{A}^1 -nilpotent sheaves, thereby verifying Theorems A and C and Corollary E.

Finally, Section 9 proves Theorem I, an étale analog of Morel’s theorem on 1-strictly and strictly \mathbb{A}^1 -invariant sheaves of abelian groups.

Furthermore, in Appendix A, we collect some results about Moore–Postnikov towers for nilpotent morphisms in the setting of ∞ -topoi.

Notation

This article is written in the language of ∞ -categories, as developed e.g. by Lurie in [Lur09]. We will use the word *anima* for an object of the ∞ -category of anima/spaces/homotopy types, whereas we will use the word *sheaf* for an object of an ∞ -topos \mathcal{X} . We will furthermore employ the following notation:

$\mathcal{A}n$	∞ -category of anima	
Sp	∞ -category of spectra	[Lur17, §1.4.3]
$\mathrm{Sp}(-)$	Stabilization of an ∞ -category	[Lur17, §1.4.2]
$\mathrm{Shv}_\tau(\mathcal{C})$	∞ -topos of sheaves on a site (\mathcal{C}, τ)	
$\mathrm{Shv}_\tau^h(\mathcal{C})$	∞ -topos of hypersheaves on a site (\mathcal{C}, τ) , i.e. the hypercompletion of $\mathrm{Shv}_\tau(\mathcal{C})$	
$\mathcal{A}b$	Category of abelian groups	
$L_{\mathbb{Q}}$	(Un)stable rationalization functor	[Mat24a, §§2, 3]
L_p	(Un)stable p -completion functor	[Mat24b, §§2, 3]
\mathcal{C}_p^\wedge	p -complete objects in an ∞ -category \mathcal{C} , i.e. the essential image of L_p	[Mat24b, §§2, 3]
Sm_S	Category of quasicompact smooth S -schemes	
$S_{\mathrm{\acute{e}t}}$	Category of quasicompact étale S -schemes	
$\mathrm{Spc}(S)$	∞ -category of motivic spaces	[BH17, §2.2]
$\mathcal{SH}^{S^1}(S)$	∞ -category of motivic S^1 -spectra	[Mor03, §4]
$\mathcal{SH}(S)$	∞ -category of motivic spectra	[BH17, §4.1]
$\mathcal{SH}_{\mathrm{\acute{e}t}}^{S^1}(S)$	∞ -category of étale motivic S^1 -spectra	[Bac21a, §5]
$\mathcal{SH}_{\mathrm{\acute{e}t}}(S)$	∞ -category of étale motivic spectra	[Bac21a, §5]
$\mathbf{K}_n^{\mathrm{MW}}$	n -th Milnor–Witt K-theory group	[Mor12, §2]

We will use without mention that there is a canonical equivalence

$$\mathrm{Sp}(\mathrm{Shv}_\tau(\mathcal{C})) \cong \mathrm{Shv}_\tau(\mathcal{C}, \mathrm{Sp})$$

between the stabilization of the ∞ -category of sheaves on (\mathcal{C}, τ) and the ∞ -category of sheaves of spectra on (\mathcal{C}, τ) , cf. [Lur18, Remark 1.3.2.2 and Proposition 1.3.1.7], and similarly in the hypercomplete case. In particular, an object $E \in \mathrm{Sp}(\mathrm{Shv}_\tau(\mathcal{C}))$ will be called a *sheaf of spectra*.

Recall that there are by construction adjunctions

$$\Sigma^\infty : \mathrm{Spc}(S)_* \rightleftarrows \mathcal{SH}^{S^1}(S) : \Omega_*^\infty$$

and

$$\sigma^\infty : \mathcal{SH}^{S^1}(S) \rightleftarrows \mathcal{SH}(S) : \omega^\infty.$$

We also write $\Sigma_{\mathbb{P}^1}^\infty := \sigma^\infty \Sigma^\infty$ and $\Omega_{\mathbb{P}^1,*}^\infty := \Omega_*^\infty \omega^\infty$ for the composed adjunction.

If k is a field, then we define the *exponential characteristic of k* as

$$e = \begin{cases} 1 & \text{if } \mathrm{char}(k) = 0 \\ p & \text{if } \mathrm{char}(k) = p > 0. \end{cases}$$

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2 Preliminaries

2.1 Limits of highly connected towers

Recall from [Mat24a, Definition 5.1 and Lemma 5.2] that a locally finite dimensional cover of an ∞ -topos \mathcal{X} consists of a jointly conservative collection of limit-preserving geometric morphisms $\{p_i^*: \mathcal{X} \rightarrow \mathcal{U}_i\}_{i \in I}$, such that moreover \mathcal{U}_i has enough points and is locally of homotopy dimension $\leq n_i$ for some $n_i \geq 0$ (depending on i). Moreover, any ∞ -topos that admits such a cover is automatically Postnikov-complete, cf. [Mat24a, Lemma 5.3]. Recall that a tower $(X_n)_n$ is called *locally highly connected* (subordinate to the cover $\{\mathcal{U}_i\}_i$) [Mat24a, Definition 6.1 (3)] if $(p_i^* X_n)_n$ is a *highly connected tower* in \mathcal{U}_i for all $i \in I$, i.e. all the $p_i^* X_n$ are connected, and for every $i \in I$ and $k \geq 1$ there exists an $N_{i,k} \geq 0$ such that for all $n \geq N_{i,k}$ the map $\pi_k(p_i^* X_n) \rightarrow \pi_k(p_i^* X_{N_{i,k}})$ is an isomorphism. In this section, we will show that under suitable finiteness conditions, geometric morphisms preserve limits of (locally) highly connected towers.

Proposition 2.1. *Let $f^*: \mathcal{X} \rightarrow \mathcal{Y}$ be a geometric morphism of ∞ -topoi. Suppose that \mathcal{X} is locally of homotopy dimension $\leq N$ for some N and that \mathcal{Y} admits a locally finite dimensional cover. Let $(X_n)_n$ be a highly connected tower in \mathcal{X} . Then the canonical map*

$$f^* \lim_n X_n \rightarrow \lim_n f^* X_n$$

is an equivalence.

Proof. Since \mathcal{Y} admits a locally finite dimensional cover, there exists a jointly conservative set of points \mathcal{S} of \mathcal{Y} such that each point $s^* \in \mathcal{S}$ factors through an ∞ -topos which is locally of homotopy dimension $\leq N_s$ for some N_s , cf. [Mat24a, Definition 5.1 (3)]. It thus suffices to check the equivalence after applying $s^* \in \mathcal{S}$. Note that we have a commutative diagram of the form

$$\begin{array}{ccc} s^* f^* \lim_n X_n & \longrightarrow & s^* \lim_n f^* X_n \\ & \searrow & \downarrow \\ & & \lim_n s^* f^* X_n \end{array}$$

where all maps are limit-assembly maps. Since $s^* f^*$ is a point of \mathcal{X} , the diagonal map is an equivalence by [Mat24a, Lemma 6.5]. By [Mat24a, Example 6.3], we see that $(f^* X_n)_n$ is highly connected (and thus in particular locally highly connected). Hence, since $s^* \in \mathcal{S}$, the vertical map is an equivalence by [Mat24a, Corollary 6.6]. Therefore, also the horizontal map is an equivalence, which is what we wanted to show. \square

We want a version of the above proposition where we relax the assumption that \mathcal{X} is locally of homotopy dimension $\leq N$. Instead, we only want both \mathcal{X} and \mathcal{Y} to admit locally finite dimensional covers. In order to prove such a result one needs a certain compatibility of locally finite dimensional covers which is captured by the following straightforward definition.

Definition 2.2. Let \mathcal{X} be an ∞ -topos with a locally finite dimensional cover $\{p_i^*: \mathcal{X} \rightarrow \mathcal{U}_i\}_{i \in I}$, and similarly \mathcal{Y} be an ∞ -topos with a locally finite dimensional cover $\{q_j^*: \mathcal{Y} \rightarrow \mathcal{V}_j\}_{j \in J}$.

A morphism of ∞ -topoi with locally finite dimensional covers consists of a geometric morphism $f^*: \mathcal{X} \rightarrow \mathcal{Y}$, a map $\kappa: J \rightarrow I$, and geometric morphisms $f_j^*: \mathcal{U}_{\kappa j} \rightarrow \mathcal{V}_j$, together with the datum of a commutative square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f^*} & \mathcal{Y} \\ \downarrow p_{\kappa j}^* & & \downarrow q_j^* \\ \mathcal{U}_{\kappa j} & \xrightarrow{f_j^*} & \mathcal{V}_j \end{array}$$

for every $j \in J$.

Proposition 2.3. *Let \mathcal{X} be an ∞ -topos with a locally finite dimensional cover $\{p_i^*: \mathcal{X} \rightarrow \mathcal{U}_i\}_{i \in I}$, and similarly \mathcal{Y} be an ∞ -topos with a locally finite dimensional cover $\{q_j^*: \mathcal{Y} \rightarrow \mathcal{V}_j\}_{j \in J}$. Let $(f^*: \mathcal{X} \rightarrow \mathcal{Y}, \kappa: J \rightarrow I, (f_j^*: \mathcal{U}_{\kappa j} \rightarrow \mathcal{V}_j)_j)$ be a morphism of ∞ -topoi with locally finite dimensional covers.*

If $(X_n)_n$ is a locally highly connected tower in \mathcal{X} (subordinate to $\{\mathcal{U}_i\}_i$), then the canonical map

$$f^* \lim_n X_n \rightarrow \lim_n f^* X_n.$$

is an equivalence.

Proof. Since the q_j^* are jointly conservative, it suffices to show that the canonical map $q_j^* f^* \lim_n X_n \rightarrow q_j^* \lim_n f^* X_n$ is an equivalence for all $j \in J$. So fix $j \in J$ and consider the following commutative diagram:

$$\begin{array}{ccc} q_j^* f^* \lim_n X_n & \xrightarrow{\sim} & f_j^* p_{\kappa j}^* \lim_n X_n \\ \downarrow \alpha & & \downarrow \gamma \\ q_j^* \lim_n f^* X_n & & f_j^* \lim_n p_{\kappa j}^* X_n \\ \downarrow \beta & & \downarrow \delta \\ \lim_n q_j^* f^* X_n & \xrightarrow{\sim} & \lim_n f_j^* p_{\kappa j}^* X_n \end{array}$$

The two horizontal arrows are given by the morphism of ∞ -topoi with finite dimensional covers, and are thus equivalences by definition. All the other maps are given by limit-assembly maps. The maps β and γ are equivalences since $p_{\kappa j}^*$ and q_j^* commute with all limits by definition. Moreover, the map δ is an equivalence by Proposition 2.1, since $(p_{\kappa j}^* X_n)_n$ is highly connected and $\mathcal{U}_{\kappa j}$ is locally of homotopy dimension $\leq N$ for some N by assumption. Hence, also the map α is an equivalence which is precisely what we wanted to show. \square

Lemma 2.4. *Let \mathcal{X} be an ∞ -topos locally of homotopy dimension $\leq N$ with enough points, and $(X_n)_n$ be a highly connected tower in \mathcal{X} . Write $X := \lim_n X_n$. Then for all $k \geq 0$ the canonical map $\pi_k(X) \rightarrow \pi_k(X_n)$ is an isomorphism for all $n \gg 0$.*

Proof. Let $k \geq 0$. By assumption, there exists N such that $\pi_k(X_n) \cong \pi_k(X_N)$ for all $n \geq N$. We will show that $\pi_k(X) \cong \pi_k(X_n)$ for all $n \geq N$. Since there are enough points, and since $s^* \lim_n X_n \cong \lim_n s^* X_n$ for all points s^* by [Mat24a, Lemma 6.5], and $s^* \pi_k(-) \cong \pi_k(s^* -)$ since s^* is a geometric morphism, we may assume that $\mathcal{X} \cong \mathcal{A}_n$. The result is now an immediate consequence of the computation of homotopy groups of the limit as e.g. done in [MP11, Proposition 2.2.9], as the system $(\pi_{k+1}(X_n))_n$ is Mittag-Leffler (it consists of isomorphisms for $n \gg 0$). \square

2.2 Unstable p -completion

Let \mathcal{X} be a Postnikov-complete ∞ -topos with enough points. Write $L_p: \mathcal{X} \rightarrow \mathcal{X}$ for the unstable p -completion functor, the Bousfield localization at the class of p -equivalences, cf. [Mat24b, Section 3], and \mathcal{X}_p^\wedge for its essential image, i.e. the subcategory of p -complete sheaves. The goal of this section is to strengthen some of the results about L_p from [Mat24b, Section 3] and [Mat24a, Section 6].

Lemma 2.5. *Let $f: E \rightarrow F$ be a morphism in $\mathrm{Sp}(\mathcal{X})$ between 1-connective sheaves of spectra. Then f is a p -equivalence if and only if $\Omega_*^\infty f$ is a p -equivalence.*

Proof. If f is a p -equivalence, then $\Omega_*^\infty f$ is a p -equivalence by [Mat24b, Lemma 3.16]. Suppose on the other hand that $\Omega_*^\infty f$ is a p -equivalence. Hence, we see that $L_p \Omega_*^\infty f: L_p \Omega_*^\infty E \rightarrow L_p \Omega_*^\infty F$ is an equivalence. Note that by [Mat24b, Lemma 3.17] this morphism is equivalent to the morphism

$$\Omega_*^\infty \tau_{\geq 1} L_p E \rightarrow \Omega_*^\infty \tau_{\geq 1} L_p F.$$

Since Ω_*^∞ is conservative on connective objects (this can e.g. be checked on stalks, where it reduces to the same claim about $\Omega_*^\infty: \mathrm{Sp}_{\geq 0} \rightarrow \mathcal{A}n_*$, which is conservative by [Lur17, Corollary 5.2.6.27]), it follows that $\tau_{\geq 1} L_p E \rightarrow \tau_{\geq 1} L_p F$ is an equivalence. Consider the canonical morphism of fiber sequences

$$\begin{array}{ccccc} \tau_{\geq 1} L_p E & \longrightarrow & L_p E & \longrightarrow & \tau_{\leq 0} L_p E \\ \downarrow \cong & & \downarrow L_p f & & \downarrow \\ \tau_{\geq 1} L_p F & \longrightarrow & L_p F & \longrightarrow & \tau_{\leq 0} L_p F. \end{array}$$

As L_p is an exact functor, p -completing again yields the morphism of fiber sequences

$$\begin{array}{ccccc} L_p \tau_{\geq 1} L_p E & \longrightarrow & L_p E & \longrightarrow & L_p \tau_{\leq 0} L_p E \\ \downarrow \cong & & \downarrow L_p f & & \downarrow \\ L_p \tau_{\geq 1} L_p F & \longrightarrow & L_p F & \longrightarrow & L_p \tau_{\leq 0} L_p F. \end{array}$$

In order to see that $f: E \rightarrow F$ is a p -equivalence, we have to show that $L_p f$ is an equivalence. By the above morphism of fiber sequences, it suffices to show that $L_p \tau_{\leq 0} L_p E \rightarrow L_p \tau_{\leq 0} L_p F$ is an equivalence. We will in fact show that both objects are equivalent to 0. We will show that $L_p \tau_{\leq 0} L_p E \cong 0$, the proof for F is the same. Since E is 1-connective, we see that $\pi_n(\tau_{\leq 0} L_p E)$ is uniquely p -divisible for all n (for $n > 0$ the homotopy object is 0, and for $n \leq 0$ use [Mat24b, Lemma 2.9] applied to the p -equivalence $E \rightarrow L_p E$). Hence, the claim follows immediately from [Mat24b, Lemma 2.10], as the standard t -structure on $\mathrm{Sp}(\mathcal{X})$ is left-separated (cf. e.g. the first paragraph of [Mat24b, Section 3.2], using that \mathcal{X} is hypercomplete). This proves the lemma. \square

Next, we strengthen [Mat24b, Proposition 3.19] to arbitrary maps of nilpotent sheaves $X \rightarrow Y$ in \mathcal{X}_* , i.e. we remove the assumption that the p -completions of X and Y are still nilpotent. For this, we need the following cofinality result:

Lemma 2.6. *Let $f: X \rightarrow Y$ be a morphism in \mathcal{X}_* of pointed connected sheaves. Then the \mathbb{N} -indexed inverse systems $(\tau_{\leq n} \text{fib}(X \rightarrow Y))_n$ and $(\text{fib}(\tau_{\leq n} X \rightarrow \tau_{\leq n} Y))_n$ are cofinal, i.e. for every n there exists morphisms*

$$\alpha_n: \text{fib}(\tau_{\leq n} X \rightarrow \tau_{\leq n} Y) \rightarrow \tau_{\leq n-1} \text{fib}(X \rightarrow Y)$$

and

$$\beta_n: \tau_{\leq n} \text{fib}(X \rightarrow Y) \rightarrow \text{fib}(\tau_{\leq n} X \rightarrow \tau_{\leq n} Y)$$

such that the compositions $\beta_{n-1}\alpha_n$ and $\alpha_n\beta_n$ are just the transition morphisms on the respective towers. In particular, the towers have equivalent limits.

Proof. A long exact sequence argument shows that the canonical map

$$\tau_{\leq n-1} \text{fib}(X \rightarrow Y) \rightarrow \tau_{\leq n-1} \text{fib}(\tau_{\leq n} X \rightarrow \tau_{\leq n} Y)$$

is an equivalence for all n . Thus, it suffices to show that the two towers $(\tau_{\leq n-1} \text{fib}(\tau_{\leq n} X \rightarrow \tau_{\leq n} Y))_n$ and $(\text{fib}(\tau_{\leq n} X \rightarrow \tau_{\leq n} Y))_n$ are cofinal.

Note first that for every n the unit gives a morphism

$$\alpha_n: \text{fib}(\tau_{\leq n} X \rightarrow \tau_{\leq n} Y) \rightarrow \tau_{\leq n-1} \text{fib}(\tau_{\leq n} X \rightarrow \tau_{\leq n} Y).$$

On the other hand, $\text{fib}(\tau_{\leq n} X \rightarrow \tau_{\leq n} Y)$ is n -truncated (as a limit of n -truncated objects), and hence the map $\text{fib}(\tau_{\leq n+1} X \rightarrow \tau_{\leq n+1} Y) \rightarrow \text{fib}(\tau_{\leq n} X \rightarrow \tau_{\leq n} Y)$ induced by the respective units $\tau_{\leq n+1} X \rightarrow \tau_{\leq n} X$ and $\tau_{\leq n+1} Y \rightarrow \tau_{\leq n} Y$ factors over a map

$$\beta_n: \tau_{\leq n} \text{fib}(\tau_{\leq n+1} X \rightarrow \tau_{\leq n+1} Y) \rightarrow \text{fib}(\tau_{\leq n} X \rightarrow \tau_{\leq n} Y).$$

It is now easy (but tedious) to see that for every n the compositions $\beta_{n-1}\alpha_n$ and $\alpha_n\beta_n$ are just the transition morphisms on the respective towers. \square

Lemma 2.7. *Let $f: X \rightarrow Y$ be a morphism in \mathcal{X}_* of pointed nilpotent sheaves. Suppose that X and Y are n -truncated for some $n \geq 0$. Then*

$$L_p \tau_{\geq 1} \text{fib}(X \rightarrow Y) \cong \tau_{\geq 1} \text{fib}(L_p X \rightarrow L_p Y).$$

Proof. Since X and Y are n -truncated, it follows from [Mat24b, Proposition 3.20] that also $L_p X$ and $L_p Y$ are nilpotent. Thus, the lemma is an immediate consequence of [Mat24b, Proposition 3.19]. \square

Proposition 2.8 (Bousfield–Kan Fiber Lemma). *Suppose that \mathcal{X} has a locally finite dimensional cover. Let $f: X \rightarrow Y$ be a morphism in \mathcal{X}_* of pointed nilpotent sheaves. Then*

$$L_p \tau_{\geq 1} \text{fib}(X \rightarrow Y) \cong \tau_{\geq 1} \text{fib}(L_p X \rightarrow L_p Y).$$

Proof. Since \mathcal{X} has a locally finite dimensional cover, we have equivalences

$$L_p X \cong \lim_n L_p \tau_{\leq n} X$$

and

$$L_p Y \cong \lim_n L_p \tau_{\leq n} Y,$$

see [Mat24a, Proposition 6.12]. Hence, we compute

$$\begin{aligned}
\tau_{\geq 1} \text{fib}(L_p X \rightarrow L_p Y) &\cong \tau_{\geq 1} \text{fib}(\lim_n L_p \tau_{\leq n} X \rightarrow \lim_n L_p \tau_{\leq n} Y) \\
&\cong \tau_{\geq 1} \lim_n \text{fib}(L_p \tau_{\leq n} X \rightarrow L_p \tau_{\leq n} Y) \\
&\cong \tau_{\geq 1} \lim_n \tau_{\geq 1} \text{fib}(L_p \tau_{\leq n} X \rightarrow L_p \tau_{\leq n} Y) \\
&\cong \tau_{\geq 1} \lim_n L_p \tau_{\geq 1} \text{fib}(\tau_{\leq n} X \rightarrow \tau_{\leq n} Y) \\
&\cong \tau_{\geq 1} \lim_n L_p \tau_{\geq 1} \tau_{\leq n} \text{fib}(X \rightarrow Y) \\
&\cong L_p \tau_{\geq 1} \text{fib}(X \rightarrow Y).
\end{aligned}$$

Here, we used that limits commute with limits in the second equivalence, e.g. [Mat24a, Lemma 4.2] in the third equivalence, Lemma 2.7 in the fourth equivalence, Lemma 2.6 in the fifth equivalence, and again [Mat24a, Proposition 6.12] and the fact that the p -completion of a connected sheaf is connected, cf. [Mat24b, Lemma 3.12], in the last equivalence. This proves the lemma. \square

Remark 2.9. Note that in the proof of Proposition 2.8, we cannot argue as in Lemma 2.7, because it is unclear (and probably wrong in general) that $L_p X$ is nilpotent for every nilpotent sheaf $X \in \mathcal{X}_*$.

In the last part of this section, our goal is a strengthened version of [Mat24a, Proposition 6.12]: We want to prove that p -completion commutes with limits along locally highly connected towers, and not just the tower of truncations $(\tau_{\leq n} X)_n$. For this, we need the following well-known lemma for which we could not find a reference in the language of ∞ -categories.

Lemma 2.10. *Let \mathcal{C} be a pointed ∞ -category with finite limits, and $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be morphisms in \mathcal{C} . Then there is a canonical fiber sequence*

$$\text{fib}(f) \rightarrow \text{fib}(gf) \rightarrow \text{fib}(g).$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\text{fib}(f) & \longrightarrow & * & & \\
\downarrow & & \downarrow & & \\
\text{fib}(gf) & \longrightarrow & \text{fib}(g) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z.
\end{array}$$

We want to show that the upper left square is cartesian. By definition, the horizontal rectangle, the vertical rectangle and the lower right square are all cartesian. Hence, we conclude by applying the pasting law for pullback squares twice (cf. the dual of [Lur09, Lemma 4.4.2.1]). \square

The next (very technical) lemma states that p -completion reduces the connectivity of a morphism between nilpotent sheaves roughly by the local homotopy dimension of the topos.

Lemma 2.11. *Suppose that \mathcal{X} is locally of homotopy dimension $\leq N$. Let $f: X \rightarrow Y \in \mathcal{X}_*$ be a morphism between nilpotent sheaves such that $\text{fib}(f)$ is $(N + k + 2)$ -connective for some $k \geq 1$. Then the fiber $\text{fib}(L_p f)$ is k -connective.*

Proof. By Corollary A.4, also the morphism f is nilpotent (cf. Definition A.1 for a definition). Hence, using Proposition A.5, we may choose a Moore–Postnikov refinement of f , i.e. a sequence of connected sheaves $(X_n)_n$ under X and over Y with $X \cong \lim_n X_n$, $X_0 \cong Y$, and fiber sequences $X_n \rightarrow X_{n-1} \rightarrow K(A_n, k_n)$, where the $A_n \in \mathcal{A}b(\text{Disc}(\mathcal{X}))$ are sheaves of abelian groups, and the $k_n \geq 2$ are integers, such that $k_n \rightarrow \infty$ as $n \rightarrow \infty$. Since $\text{fib}(f)$ is $(N + k + 2)$ -connective, we may even assume that $k_n \geq N + k + 2$ for all n , cf. Lemma A.6. Note that since Y is nilpotent, using [Mat24b, Lemma A.12] inductively, we conclude that also all the X_n are nilpotent. Moreover, it is clear that $(X_n)_n$ is a highly connected tower. By Proposition 2.8 we see that $L_p X_n \cong \tau_{\geq 1} \text{fib}(L_p X_{n-1} \rightarrow L_p K(A_n, k_n))$. Using the proof of [Mat24a, Lemma 6.11] (where the connectivity of the p -completion of an Eilenberg–MacLane sheaf is computed), we see that $L_p K(A_n, k_n)$ is at least $(k + 1)$ -connective, and the connectivity tends to infinity as $n \rightarrow \infty$. As $L_p X_{n-1}$ is connected, we see by a long exact sequence argument that already $\text{fib}(L_p X_{n-1} \rightarrow L_p K(A_n, k_n))$ is connected, and hence $L_p X_n \cong \text{fib}(L_p X_{n-1} \rightarrow L_p K(A_n, k_n))$. Hence, $\text{fib}(L_p X_n \rightarrow L_p X_{n-1}) \cong \Omega L_p K(A_n, k_n)$ is at least k -connective, again with connectivity tending to infinity as $n \rightarrow \infty$. By Lemma 2.10 we have a fiber sequence

$$\text{fib}(L_p X_{n+1} \rightarrow L_p X_n) \rightarrow \text{fib}(L_p X_{n+1} \rightarrow L_p Y) \rightarrow \text{fib}(L_p X_n \rightarrow L_p Y).$$

Since $X_0 \cong Y$, we can thus inductively prove that also $\text{fib}(L_p X_{n+1} \rightarrow L_p Y)$ is at least k -connective. Moreover, as the connectivity of $\text{fib}(L_p X_{n+1} \rightarrow L_p X_n)$ tends to infinity as $n \rightarrow \infty$, we conclude that in fact $(\text{fib}(L_p X_n \rightarrow L_p Y))_n$ is a highly connected tower. Additionally, we can compute its limit:

$$\lim_n \text{fib}(L_p X_n \rightarrow L_p Y) \cong \text{fib}(\lim_n L_p X_n \rightarrow L_p Y) \cong \text{fib}(L_p X \rightarrow L_p Y),$$

where the second equivalence is Proposition 2.13, using that $(X_n)_n$ is a highly connected tower of nilpotent sheaves. Hence, we conclude from Lemma 2.4 that also $\text{fib}(L_p X \rightarrow L_p Y)$ is k -connective, which is exactly what we wanted to prove. \square

We also need the following result which is a strengthening of [Mat24a, Lemma 6.11]:

Lemma 2.12. *Suppose that \mathcal{X} admits a locally finite dimensional cover. Let $(X_n)_n$ be a locally highly connected tower (subordinate to this cover), such that all the sheaves $X_n \in \mathcal{X}_*$ are nilpotent. Then also the tower $(L_p X_n)_n$ is locally highly connected.*

Proof. Since p_i^* commutes with the p -completion by [Mat24a, Lemma 6.10], and preserves nilpotent objects (as it is the left adjoint of a geometric morphism), we can assume that \mathcal{X} is locally of homotopy dimension $\leq N$ for some $N \in \mathbb{N}$ and that $(X_n)_n$ is a highly connected tower, and our goal is to show that also $(L_p X_n)_n$ is highly connected. So let $k \in \mathbb{N}$. Since $(X_n)_n$ is highly connected, there exists $L \geq 0$ such that for all $m \geq L$ the fiber $\text{fib}(X_m \rightarrow X_L)$ is $(k + N + 3)$ -connective. We have to find $M \geq 0$ such that for all $m \geq M$ the fiber $\text{fib}(L_p X_m \rightarrow L_p X_M)$ is $(k + 1)$ -connective. The claim follows from Lemma 2.11 using $M := L$. \square

Proposition 2.13. *Suppose that \mathcal{X} admits a locally finite dimensional cover. Let $(X_n)_n$ be a locally highly connected tower (subordinate to this cover), such that all the sheaves $X_n \in \mathcal{X}_*$ are nilpotent. Then the canonical map $L_p \lim_n X_n \rightarrow \lim_n L_p X_n$ is an equivalence.*

Proof. The proof is identical to that of [Mat24a, Proposition 6.12], but we now need Lemma 2.12 to see that $(L_p X_n)_n$ is still locally highly connected. \square

2.3 \mathbb{P}^1 -Postnikov towers

In the sequel we will need that over a perfect field any 2-effective nilpotent motivic space admits a version of the Postnikov-tower, where the layers are infinite \mathbb{P}^1 -loop spaces. This is a deep result of Asok–Bachmann–Hopkins [ABH23], which we now recall:

Proposition 2.14 (Asok–Bachmann–Hopkins). *Let k be a perfect field and $X \in \mathrm{Spc}(k)_*$ a pointed nilpotent motivic space. Suppose moreover that X is q -effective for some $q \geq 2$. Then X admits a refined Postnikov tower where the layers are q -effective infinite loop spaces, i.e. there exists an \mathbb{N} -indexed inverse system $X_\bullet: \mathbb{N}^{\mathrm{op}} \rightarrow \mathrm{Spc}(k)_{*,X/}$ of pointed motivic spaces X_i under X , and fiber sequences $X_{i+1} \rightarrow X_i \rightarrow K_i$, such that*

- (a) $X_0 \cong *$,
- (b) for every i , X_i is nilpotent and q -effective,
- (c) $X \cong \lim_i X_i$,
- (d) the connectivities of the K_i tend to ∞ as $i \rightarrow \infty$, and
- (e) for every i , there exists a motivic spectrum $E_i \in \mathcal{SH}(k)$ which is 2-connective and q -effective, and an equivalence $K_i \cong \Omega_{\mathbb{P}^1,*}^\infty E_i$. In particular K_i is 2-connective and q -effective.

Proof. It follows from [ABH23, Construction 4.1.7] that there exist motivic spaces X_i under X and fiber sequences $X_{i+1} \rightarrow X_i \rightarrow K_i$, satisfying (a), (b), (c) and (d), such that $K_i \cong \tau_{\geq(q+1,q)} K(A_i, n_i)$ with A_i a strictly \mathbb{A}^1 -invariant sheaf of abelian groups, $n_i \geq 2$ and $n_i \rightarrow \infty$ as $i \rightarrow \infty$. Here we use the notation $\tau_{\geq(q+1,q)}$ from [ABH23, Definition 4.1.2]. Now the remaining claim (e) is [ABH23, Remark 4.1.13]. \square

Remark 2.15. In the sequel we will repeatedly use some kind of *induction* on this tower. The main strategy will be the following: If a statement for motivic spaces is stable under limits and holds for sheaves of the form $\Omega_{\mathbb{P}^1,*}^\infty G$ for $G \in \mathcal{SH}(k)$, then it holds for any nilpotent 2-effective motivic space.

This version of the Postnikov-tower behaves well with respect to rationalization, as captured by the following lemma.

Lemma 2.16. *Let k be a perfect field and $X \in \mathrm{Spc}(k)_*$ a pointed nilpotent motivic space. Suppose moreover that X is q -effective, where $q \geq 2$. Let $R \subseteq \mathbb{Q}$ be a subring. If X is R -local, then we may assume that also the X_i , K_i and E_i from Proposition 2.14 are R -local.*

Proof. Choose sheaves X_i , K_i and E_i for all i as in Proposition 2.14. We show that also the collection of $L_R X_i$, $L_R K_i$ and $L_R E_i$ satisfies (a) to (e) of *loc. cit.*.

First, it follows from [AFH22, Theorem 4.3.11] that $L_R X_{i+1} \rightarrow L_R X_i \rightarrow L_R K_i$ is still a fiber sequence. We prove the remaining points:

(a): It is clear that $L_R X_0 \cong L_R * \cong *$.

(b): That $L_R X_i$ is nilpotent was shown in [AFH22, Proposition 4.3.8] (they only show that $L_R X_i$ is weakly \mathbb{A}^1 -nilpotent, but under the assumption that X_i is nilpotent their proof in fact shows that $L_R X_i$ is nilpotent). That it is q -effective was shown in [ABH23, Proposition 4.3.4].

(c): We have $X \cong L_R X \cong L_R \lim_n X_n \cong \lim_n L_R X_n$ by combining [AFH22, Proposition 4.3.8 and Theorem 4.3.9] (which shows that the R -localization of a motivic space can be computed as the R -localization of the underlying Nisnevich sheaf) with [Mat24a, Proposition 6.9] (which applies as $\mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_k)$ admits a locally finite dimensional cover, cf. [Mat24a, Proposition A.3]).

(d): This follows from [AFH22, Proposition 4.3.8].

(e): It was shown in [ABH23, Lemma 3.2.7 (1) and Corollary 3.2.9] that $L_R E_i$ is still 2-connective and q -effective. Note that $L_R K_i \cong L_R \Omega_{\mathbb{P}^1, *}^\infty E_i \cong \Omega_*^\infty L_R E_i$. Indeed, we have $\Omega_{\mathbb{P}^1, *}^\infty = \Omega_*^\infty \omega^\infty$, and Ω_*^∞ commutes with L_R on 1-connective objects by [Mat24a, Lemma 3.18] (or rather its analog for motivic spaces, which can be proven in exactly the same way), whereas ω^∞ commutes with L_R by [Mat24a, Corollary 2.7] as it preserves filtered colimits by [BY20, Lemma 6.1]. \square

2.4 Étale ∞ -topoi

In this section, we collect some results about the small-étale and smooth-étale ∞ -topoi.

Definition 2.17. Let S be a qcqs scheme.

- We write $S_{\mathrm{\acute{e}t}}$ for the category of qcqs étale S -schemes, equipped with the étale topology, and $\mathrm{Shv}_{\mathrm{\acute{e}t}}^{\mathrm{h}}(S_{\mathrm{\acute{e}t}})$ for the ∞ -topos of étale hypersheaves on $S_{\mathrm{\acute{e}t}}$.
- We write Sm_S for the category of qcqs smooth S -schemes, equipped with the étale topology, and $\mathrm{Shv}_{\mathrm{\acute{e}t}}^{\mathrm{h}}(\mathrm{Sm}_S)$ for the ∞ -topos of étale hypersheaves on Sm_S .

Proposition 2.18. *Let S be a scheme. A conservative family of points of $\mathrm{Shv}_{\mathrm{\acute{e}t}}^{\mathrm{h}}(\mathrm{Sm}_S)$ is given by evaluating at $\mathrm{Spec}(\mathcal{O}_{X, \bar{x}}^{\mathrm{sh}})$, the spectrum of the strict henselization of the local ring of X at \bar{x} , where X is a smooth S -scheme and $\bar{x} \in X$ is a geometric point.*

Similarly, a conservative family of points of $\mathrm{Shv}_{\mathrm{\acute{e}t}}^{\mathrm{h}}(S_{\mathrm{\acute{e}t}})$ is given by evaluating at $\mathrm{Spec}(\mathcal{O}_{S, \bar{s}}^{\mathrm{sh}})$, the spectrum of the strict henselization of the local ring of S at \bar{s} , where $\bar{s} \in S$ is a geometric point.

Proof. That these functors are points is [CM21, Example 4.32]. We now prove that they are jointly conservative. Since we work with hypersheaves, we can check equivalences on homotopy sheaves and are thus reduced to show that on the underlying 1-topos those points form a conservative family. This was proven in [Sta23, Theorem 03PU] for the small étale site, the smooth case

follows similarly (e.g. by pulling back to various small étale sites $\mathrm{Shv}_{\mathrm{ét}}^h(X_{\mathrm{ét}})$ for $X \in \mathrm{Sm}_S$). \square

Definition 2.19. Let S be a qcqs scheme, and $x \in S$. Write $\mathrm{cd}(x) \in \mathbb{N} \cup \{\infty\}$ for the étale cohomological dimension of the field $k(x)$, cf. [Sta23, Tag 0F0Q]. Similarly, write $\mathrm{cd}(S) \in \mathbb{N} \cup \{\infty\}$ for the étale cohomological dimension of S .

Definition 2.20. Let S be a qcqs scheme. We say that S is *étale bounded* if there is a global bound on the étale cohomological dimension of the residue fields of S , i.e. $\sup_{x \in S} \mathrm{cd}(x) < \infty$.

Lemma 2.21. *Let S be a qcqs scheme of finite Krull-dimension that is étale bounded. Then also $\mathrm{cd}(S) < \infty$. Moreover, if $\sup_{s \in S} \mathrm{cd}(s) \leq N$, then $\mathrm{cd}(S) \leq N + \dim(S)$.*

Proof. See e.g. the proof of [CM21, Corollary 3.29]. \square

Lemma 2.22. *Let S be a scheme, and $U \rightarrow S$ an étale S scheme. The canonical functor*

$$\mathrm{Shv}_{\mathrm{ét}}^h(S_{\mathrm{ét}}) \rightarrow \mathrm{Shv}_{\mathrm{ét}}^h(U_{\mathrm{ét}})$$

induces an equivalence

$$\mathrm{Shv}_{\mathrm{ét}}^h(S_{\mathrm{ét}})_{/U} \cong \mathrm{Shv}_{\mathrm{ét}}^h(U_{\mathrm{ét}})$$

Proof. Since morphisms between étale S -schemes are themselves étale [Sta23, Tag 02GW], we get an equivalence of sites $(S_{\mathrm{ét}})_{/U} \cong U_{\mathrm{ét}}$, where the slice category carries the canonical site structure. Then the result follows from [GAV71, Exposé III, Proposition 5.4], as all involved ∞ -topoi are (hypercompletions of) 1-localic topoi. \square

Lemma 2.23. *Let S be a qcqs scheme of finite Krull-dimension, such that S is étale bounded. There exists an $N \geq 0$ such that $\mathrm{Shv}_{\mathrm{ét}}^h(S_{\mathrm{ét}})$ is locally of homotopy dimension $\leq N$. In particular, it is Postnikov-complete.*

Proof. Let $N := \dim(S) + \sup_{x \in S} \mathrm{cd}(x)$. This is finite since S is étale bounded. By [CM21, Corollary 3.29] we see that for every étale S -scheme $U \rightarrow S$ the cohomological dimension of $\mathrm{Shv}_{\mathrm{ét}}^h(U_{\mathrm{ét}})$ is $\leq N$. Note that $\mathrm{Shv}_{\mathrm{ét}}^h(S_{\mathrm{ét}})_{/U} \cong \mathrm{Shv}_{\mathrm{ét}}^h(U_{\mathrm{ét}})$ by Lemma 2.22. In particular, we see that $\mathrm{Shv}_{\mathrm{ét}}^h(S_{\mathrm{ét}})$ is locally of cohomological dimension $\leq N$. Since it is hypercomplete by definition, it follows that it is locally of homotopy dimension $\leq N$, cf. [Lur18, Proposition 1.3.3.10]. It follows from [Lur09, Proposition 7.2.1.10] that it is also Postnikov-complete. \square

Lemma 2.24. *Let S be a qcqs scheme of finite Krull-dimension, which is moreover étale bounded. Let $p: X \rightarrow S$ be a smooth finite type S -scheme. Then X is étale bounded.*

Proof. Let $K := \sup_{s \in S} \mathrm{cd}(s)$, and let $M := \sup_{x \in X} \mathrm{trdeg}(k(x)/k(p(x)))$ be the supremum over the transcendence degrees. K is finite by assumption, and by combining [Sta23, Tag 0A3V] with [Sta23, Tag 0A21] we see that also M is finite. We claim that $\sup_{x \in X} \mathrm{cd}(x) \leq K + M$. This follows from [Sta23, Tag 0F0T] as for $x \in X$ we have $\mathrm{cd}(x) \leq \mathrm{cd}(p(x)) + \mathrm{trdeg}(k(x)/k(p(x))) \leq K + M$. \square

Proposition 2.25. *Let S be a qcqs scheme of finite Krull-dimension, which is moreover étale bounded. Then $\mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S)$ admits a locally finite dimensional cover in the sense of [Mat24a, Definition 5.1]. In particular, it is Postnikov-complete.*

Proof. Arguing as in the proof of [Mat24a, Proposition A.3], one reduces to the claim that for $U \in \mathrm{Sm}_S$, the ∞ -topos $\mathrm{Shv}_{\mathrm{ét}}^h(U_{\mathrm{ét}})$ is locally of homotopy dimension $\leq N$ for some N and has enough points. The first statement was proven in Lemma 2.23, as U is étale bounded, cf. Lemma 2.24. The other statement is Proposition 2.18. That $\mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S)$ is Postnikov-complete follows from [Mat24a, Lemma 5.3]. \square

Proposition 2.26. *Let S be a qcqs scheme of finite Krull-dimension, which is moreover étale bounded. The étale hypersheafification functor $L_{\mathrm{ét}} : \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_S) \rightarrow \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S)$ upgrades to a morphism of ∞ -topoi with locally finite dimensional covers, where we equip the Nisnevich topos with the locally finite dimensional cover from [Mat24a, Proposition A.3], and the étale topos with the cover from Proposition 2.25.*

Proof. For this, it suffices to note that étale hypersheafification commutes with restriction to a small étale site, i.e. that for every smooth S -scheme U the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_S) & \longrightarrow & \mathrm{Shv}_{\mathrm{nis}}(U_{\mathrm{ét}}) \\ \downarrow L_{\mathrm{ét}} & & \downarrow L_{\mathrm{ét}} \\ \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S) & \longrightarrow & \mathrm{Shv}_{\mathrm{ét}}^h(U_{\mathrm{ét}}). \end{array}$$

This follows as already the corresponding diagram of sites commutes. \square

Definition 2.27. We write $\iota : S_{\mathrm{ét}} \rightarrow \mathrm{Sm}_S$ for the inclusion of sites, and

$$\iota_! : \mathrm{Shv}_{\mathrm{ét}}^h(S_{\mathrm{ét}}) \rightleftarrows \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S) : \iota^*$$

for the induced geometric morphism on the associated hypercomplete ∞ -topoi, i.e. the hypercompletion of the geometric morphism from [Pst22, Proposition A.11]. Here, ι^* is given by restriction along ι , and $\iota_!$ is left Kan extension along ι , followed by hypersheafification. Note that ι^* also has a right adjoint ι_* : It exists before the hypercompletion by [Pst22, Proposition A.13] because ι has the covering lifting property [Pst22, Definition A.12].

Lemma 2.28. *The functors $\iota_!$ and ι_* are fully faithful.*

Proof. The functor $\iota_!$ is fully faithful by [Bac21a, Lemma 6.1], note that in the reference, e^* is the name for $\iota_!$. Since ι_* is right adjoint to the right adjoint of $\iota_!$, it is formal that it is also fully faithful (cf. (the dual of) [NPR24, Corollary 2.7] for a fun proof of this fact). \square

Definition 2.29. Let $f : T \rightarrow S$ be a morphism of schemes. Pullback along f defines a morphism of sites $\mathrm{Sm}_S \rightarrow \mathrm{Sm}_T$. We write

$$f^* : \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S) \rightleftarrows \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_T) : f_*$$

for the induced geometric morphism on the associated hypercomplete ∞ -topoi, i.e. the hypercompletion of the geometric morphism from [Pst22, Proposition A.11]. Here, f_* is given by restriction along the morphism of sites, and f^* is left Kan extension along f , followed by hypersheafification.

Similarly, we have a geometric morphism

$$f^*: \mathrm{Shv}_{\acute{\mathrm{et}}}^h(S_{\acute{\mathrm{et}}}) \rightleftarrows \mathrm{Shv}_{\acute{\mathrm{et}}}^h(T_{\acute{\mathrm{et}}}) : f_*$$

using the small étale sites.

Definition 2.30. Let S be a scheme, and $\bar{s} \in S$ a geometric point. Write $\rho_{\bar{s}}: S_{\bar{s}}^{\mathrm{sh}} := \mathrm{Spec} \mathcal{O}_{S, \bar{s}}^{\mathrm{sh}} \rightarrow S$ for the canonical morphism from the spectrum of the strict henselization of the local ring of S at \bar{s} to S . In particular, using Definition 2.29 we have adjunctions

$$\rho_{\bar{s}}^*: \mathrm{Shv}_{\acute{\mathrm{et}}}^h(\mathrm{Sm}_S) \rightleftarrows \mathrm{Shv}_{\acute{\mathrm{et}}}^h(\mathrm{Sm}_{S_{\bar{s}}^{\mathrm{sh}}}) : \rho_{\bar{s},*}$$

and

$$\rho_{\bar{s}}^*: \mathrm{Shv}_{\acute{\mathrm{et}}}^h(S_{\acute{\mathrm{et}}}) \rightleftarrows \mathrm{Shv}_{\acute{\mathrm{et}}}^h((S_{\bar{s}}^{\mathrm{sh}})_{\acute{\mathrm{et}}}) : \rho_{\bar{s},*}.$$

Lemma 2.31. *Let S be a scheme and $\bar{s} \in S$ a geometric point. Suppose that $T \in \mathrm{Sm}_{S_{\bar{s}}^{\mathrm{sh}}}$. Then there exists a cofinal filtered system $(\bar{s} \rightarrow U_i \rightarrow S)_i$ of quasiconpact étale neighborhoods of \bar{s} in S (i.e. $S_{\bar{s}}^{\mathrm{sh}} \cong \lim_i U_i$), and for every i a smooth U_i -scheme T_i , together with transition morphisms $T_i \rightarrow T_j$ making the obvious diagram commute, such that $T \cong \lim_i T_i$. If T is étale over $S_{\bar{s}}^{\mathrm{sh}}$, then we can arrange that all the $T_i \rightarrow U_i$ are étale.*

Similarly, we can descend any étale cover $T' \rightarrow T$ in $\mathrm{Sm}_{S_{\bar{s}}^{\mathrm{sh}}}$, or any disjoint union decomposition $T = \coprod_j T_j$.

Proof. Combine [Sta23, Lemma 01ZM] with [Sta23, Lemma 0C0C]. For the étale statements, we instead use [Sta23, Lemma 07RP], where we additionally use descent for surjective morphisms [Sta23, Lemma 07RR] for the covers. For disjoint union decompositions, we can combine [Sta23, Lemma 01ZP] with [Sta23, Lemma 0EUU] and again the statement about surjective morphisms. \square

Lemma 2.32. *Let \mathcal{X} be an ∞ -topos, and $T \in \mathcal{X}$ an object of homotopy dimension $\leq N$ for some $N \geq 0$. Let $X \in \mathcal{X}$ and $m \geq 0$. Write $f: X \rightarrow \tau_{\leq N+m} X$ for the canonical map. Then $\mathrm{Map}_{\mathcal{X}}(T, f)$ is an $(m+1)$ -connective map of anima.*

Proof. The global sections functor $\Gamma: \mathcal{X}_T \rightarrow \mathcal{A}n$ sends k -connective morphisms to $(k-N)$ -connective morphisms, cf. [Lur09, Lemma 7.2.1.7]. The map f is $(N+m+1)$ -connective, and so is $f \times T: X \times T \rightarrow (\tau_{\leq N+m} X) \times T \in \mathcal{X}_T$. Note that $\Gamma(f \times T) \cong \mathrm{Map}_{\mathcal{X}_T}(T, f \times T) \cong \mathrm{Map}_{\mathcal{X}}(T, f)$, hence we see that $\mathrm{Map}_{\mathcal{X}}(T, f)$ is an $(m+1)$ -connective map of anima. \square

Lemma 2.33. *Let S be a qcqs scheme of finite Krull dimension which is moreover étale bounded, and $\bar{s} \in S$ a geometric point. Let $\{\bar{s} \rightarrow U_i \rightarrow S\}_{i \in I}$ be a cofinal system of étale neighborhoods of \bar{s} , and $0 \in I$ be an initial object. Suppose that $T_0 \rightarrow U_0$ is a scheme over U_0 such that $T_0 \rightarrow U_0 \rightarrow S$ is smooth. Let $T_i := T_0 \times_{U_0} U_i$ and $T := T_0 \times_{U_0} S_{\bar{s}}^{\mathrm{sh}} \cong \lim_i T_i$. Then for every $X \in \mathrm{Shv}_{\acute{\mathrm{et}}}^h(\mathrm{Sm}_S)$ we have $\rho_{\bar{s}}^* X(T) \cong \mathrm{colim}_i X(T_i)$.*

Proof. We first prove the lemma under the additional assumption that X is N -truncated for some $N \geq 0$. Write $f^*: \mathcal{P}(\mathrm{Sm}_S) \rightleftarrows \mathcal{P}(\mathrm{Sm}_{S^{\mathrm{sh}}}) : f_*$ for the adjunction on presheaves where f_* is given by precomposition along the pullback functor $f: \mathrm{Sm}_S \rightarrow \mathrm{Sm}_{S^{\mathrm{sh}}}$, and f^* is left Kan extension. In particular, $\rho_s^* X \cong L_{\mathrm{ét}} f^* X$. Hence, it suffices to show that (1) $f^* X(T) \cong \mathrm{colim}_i X(T_i)$ and (2) $f^* X$ satisfies étale hyperdescent.

(1): By definition of left Kan extension we have an equivalence

$$f^* X(T) \cong \mathrm{colim}_{U, T \rightarrow f(U)} X(U),$$

where the colimit runs over the comma category $(T \downarrow \mathrm{Sm}_S)$. By the universal property of the pullback, this category is equivalent to $(\mathrm{Sm}_S)_{T/}$, where we view T as an S -scheme. It is left to show that the T_i are cofinal in this category. For this, let $T \rightarrow U$ any S -morphism with $U \in \mathrm{Sm}_S$, and we have to show that there is an i such that $T \rightarrow U$ factors over $T \rightarrow T_i$. This is an immediate application of [Sta23, Proposition 01ZC].

(2): Since f^* preserves truncated objects, and since any truncated sheaf is automatically a hypersheaf, it suffices to show that $f^* X$ is an étale sheaf. Note that $f^* X$ is a Σ -sheaf (i.e. it sends finite coproducts to products): Indeed, let $(V_j)_j$ be a finite family of smooth schemes over S^{sh} . By Lemma 2.31 we may assume that we get compatible families $(V_{j,i})_j$ over a cofinal system of étale neighborhoods $\bar{s} \rightarrow U_i \rightarrow S$, such that $V_{j,i} \times_{U_i} S^{\mathrm{sh}} \cong V_j$. Now, by (1) we know that $f^* X(\coprod_j V_j) \cong \mathrm{colim}_i X(\coprod_j V_{j,i})$. Since X is a Σ -sheaf, and since finite limits commute with filtered colimits in $\mathcal{A}n$, we conclude that also $f^* X$ is a Σ -sheaf. Hence, it now suffices to prove the sheaf condition for an étale cover $V \rightarrow U$ consisting of a single morphism. Again, using Lemma 2.31, we can find compatible étale covers $V_i \rightarrow U_i$. We argue as above, using (1) and the fact that on homotopy groups the totalization of the Čech nerve behaves like a finite limit, and hence again commutes with filtered colimits.

We end the proof by showing the general statement. So let $X \in \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S)$ be arbitrary. First note that T_0 is étale bounded, with some bound $K \geq 0$ by Lemma 2.24. Moreover, by the proof of this lemma, since any T_i is an étale T_0 -scheme, we see that it is again étale bounded with the same bound K , and similarly for the pro-étale T_0 -scheme T . Hence, all the $T_i \in \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S)$ and also $T \in \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_{S^{\mathrm{sh}}})$ have cohomological dimension $\leq N$, where $N := K + \dim(T_0)$, cf. Lemma 2.21 (as $\dim(T) = \dim(T_i) = \dim(T_0)$). By the proof of [Lur09, Proposition 1.3.3.10], using Postnikov-completeness, T and the T_i have in fact homotopy dimension $\leq N$. We now have for every $m \geq 0$ a commutative diagram

$$\begin{array}{ccc} (\rho_s^* X)(T) = \mathrm{Map}_{S^{\mathrm{sh}}} (T, \rho_s^* X) & \longrightarrow & \mathrm{Map}_{S^{\mathrm{sh}}} (T, \tau_{\leq N+m} \rho_s^* X) \\ \downarrow & & \downarrow \cong \\ & & \mathrm{Map}_{S^{\mathrm{sh}}} (T, \rho_s^* \tau_{\leq N+m} X) \\ & & \downarrow \cong \\ \mathrm{colim}_i X(T_i) = \mathrm{colim}_i \mathrm{Map}_S (T_i, X) & \longrightarrow & \mathrm{colim}_i \mathrm{Map}_S (T_i, \tau_{\leq N+m} X) \end{array}$$

Here, the top right vertical map is an equivalence since ρ_s^* is a geometric morphism, and the bottom right vertical map is an equivalence by the above special

case where X was assumed to be truncated. Our goal is to show that the left vertical map of anima is an equivalence, it suffices to show that it is ∞ -connective. For this, we will show that the horizontal maps are both m -connective, since the left vertical map is independent of m , this then proves the claim. First note that m -connective maps are stable under filtered colimits. Since T and the T_i have homotopy dimension $\leq N$, the claim now follows from Lemma 2.32. \square

Lemma 2.34. *Let S be a qcqs scheme of finite Krull dimension, that is moreover étale bounded, and $\bar{s} \in S$ a geometric point. The following squares are commutative:*

$$\begin{array}{ccc} \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(S_{\mathrm{\acute{e}t}}) & \xrightarrow{\rho_{\bar{s}}^*} & \mathrm{Shv}_{\mathrm{\acute{e}t}}^h((S_{\bar{s}}^{\mathrm{sh}})_{\mathrm{\acute{e}t}}) \\ \downarrow \iota_! & & \downarrow \iota_! \\ \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S) & \xrightarrow{\rho_{\bar{s}}^*} & \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_{S_{\bar{s}}^{\mathrm{sh}}}) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(S_{\mathrm{\acute{e}t}}) & \xrightarrow{\rho_{\bar{s}}^*} & \mathrm{Shv}_{\mathrm{\acute{e}t}}^h((S_{\bar{s}}^{\mathrm{sh}})_{\mathrm{\acute{e}t}}) \\ \uparrow \iota^* & & \uparrow \iota^* \\ \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S) & \xrightarrow{\rho_{\bar{s}}^*} & \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_{S_{\bar{s}}^{\mathrm{sh}}}). \end{array}$$

Proof. For the first square, it suffices to show that the associated square of right adjoints commutes. As all the right adjoints are given by restriction along morphisms of sites, this follows since they already commute on the level of sites.

For the second square, note that there exists a natural transformation

$$\rho_{\bar{s}}^* \iota^* \xrightarrow[\simeq]{\mathrm{unit}} \iota^* \iota_! \rho_{\bar{s}}^* \iota^* \cong \iota^* \rho_{\bar{s}}^* \iota_! \iota^* \xrightarrow{\mathrm{counit}} \iota^* \rho_{\bar{s}}^*$$

given by the Beck–Chevalley transformation (using the commutativity of the first diagram). The morphism induced by the unit is an equivalence since $\iota_!$ is fully faithful, cf. Lemma 2.28. Hence, it suffices to show that also the morphism induced by the counit is an equivalence. So let $X \in \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)$. Since ι^* is given by restriction, we have to show that for every $T \in (S_{\bar{s}}^{\mathrm{sh}})_{\mathrm{\acute{e}t}}$ the canonical map

$$(\rho_{\bar{s}}^* \iota_! \iota^* X)(T) \xrightarrow{\mathrm{counit}} (\rho_{\bar{s}}^* X)(T)$$

is an equivalence. By Lemma 2.31 we may choose étale neighborhoods $\{\bar{s} \rightarrow U_i \rightarrow S\}_i$ and for every i an étale U_i -scheme T_i such that $T \cong \lim_i T_i$. It now follows from Lemma 2.33 that this morphism is equivalent to the morphism

$$\mathrm{colim}_i (\iota_! \iota^* X)(T_i) \xrightarrow{\mathrm{counit}} \mathrm{colim}_i X(T_i).$$

Since $T_i \in S_{\mathrm{\acute{e}t}}$ and $\iota_!$ is given by left Kan extension along the inclusion $S_{\mathrm{\acute{e}t}} \rightarrow \mathrm{Sm}_S$, it follows that this morphism is an equivalence. \square

Lemma 2.35. *Let S be a qcqs scheme of finite Krull dimension, that is moreover étale bounded. The collection of functors $\{\rho_{\bar{s}}^*\}_{\bar{s} \in S}$ is jointly conservative on both $\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)$ and $\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(S_{\mathrm{\acute{e}t}})$ (where the \bar{s} are the geometric points of S).*

Proof. We first show that the functors $\rho_{\bar{s}}^*: \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S) \rightarrow \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_{S_{\bar{s}}^{\mathrm{sh}}})$ are jointly conservative. So let $f: X \rightarrow Y$ be a morphism in $\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)$ such that $\rho_{\bar{s}}^* f$ is an equivalence for all geometric points \bar{s} . As for the étale topology strictly henselian local rings form the points of the topos, and we are working with hypersheaves, it suffices to show that for every $T \in \mathrm{Sm}_S$ and geometric point $\bar{t} \in T$ the morphism $f: X(T_{\bar{t}}^{\mathrm{sh}}) \rightarrow Y(T_{\bar{t}}^{\mathrm{sh}})$ is an equivalence. Fix T and \bar{t} . Hence, we have to show that

$$f: \operatorname{colim}_{\bar{t} \rightarrow U \xrightarrow{\mathrm{\acute{e}t}} T} X(U) \rightarrow \operatorname{colim}_{\bar{t} \rightarrow U \xrightarrow{\mathrm{\acute{e}t}} T} Y(U) \quad (1)$$

is an equivalence. Write $\bar{s} \in S$ for the geometric point under $\bar{t} \in T$, so that there is a commutative diagram

$$\begin{array}{ccc} T_{\bar{t}}^{\mathrm{sh}} & \xrightarrow{\rho_{\bar{t}}} & T \\ \downarrow \alpha & & \downarrow \beta \\ S_{\bar{s}}^{\mathrm{sh}} & \xrightarrow{\rho_{\bar{s}}} & S. \end{array}$$

In particular, $\rho_{\bar{t}}^* \beta^* f = \alpha^* \rho_{\bar{s}}^* f$ is an equivalence. By Lemma 2.33, the equivalence $\rho_{\bar{t}}^* \beta^* f: \rho_{\bar{t}}^* \beta^* X(T_{\bar{t}}^{\mathrm{sh}}) \cong \rho_{\bar{t}}^* \beta^* Y(T_{\bar{t}}^{\mathrm{sh}})$ may be identified with the morphism

$$\beta^* f: \operatorname{colim}_{\bar{t} \rightarrow U \xrightarrow{\mathrm{\acute{e}t}} T} \beta^* X(U) \xrightarrow{\sim} \operatorname{colim}_{\bar{t} \rightarrow U \xrightarrow{\mathrm{\acute{e}t}} T} \beta^* Y(U).$$

Since any étale T -scheme U is in particular a smooth S -scheme, we see that $\beta^* Z(U) \cong Z(U)$ for any Z . Hence, the above equivalence can be identified with the morphism (1), which proves the claim.

We finish the proof by showing that also the functors $\rho_{\bar{s}}^*: \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(S_{\mathrm{\acute{e}t}}) \rightarrow \mathrm{Shv}_{\mathrm{\acute{e}t}}^h((S_{\bar{s}}^{\mathrm{sh}})_{\mathrm{\acute{e}t}})$ are jointly conservative. For this, let $f: X \rightarrow Y \in \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(S_{\mathrm{\acute{e}t}})$ be a morphism such that $\rho_{\bar{s}}^* f$ is an equivalence for all geometric points $\bar{s} \in S$. Then also $\iota_! \rho_{\bar{s}}^* f \cong \rho_{\bar{s}}^* \iota_! f$ is an equivalence for all $\bar{s} \in S$, using Lemma 2.34. Hence, it follows from the first part that $\iota_! f$ is an equivalence. Thus, we deduce that f is already an equivalence since $\iota_!$ is fully faithful, cf. Lemma 2.28. \square

Lemma 2.36. *Let k be a separably closed field. Then evaluation at k induces an equivalence $\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(k_{\mathrm{\acute{e}t}}) \xrightarrow{\sim} \mathcal{A}n$.*

Proof. Since k is separably closed, the functor $\mathrm{Fin} \rightarrow k_{\mathrm{\acute{e}t}}$ from finite sets to $k_{\mathrm{\acute{e}t}}$ given by sending a set A to $\Pi_A \mathrm{Spec}(k)$ is an equivalence of categories. Under this equivalence, the étale topology corresponds to the topology generated by finite coproduct decompositions. Hence, using [BH17, Lemma 2.4], we see that $\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(k_{\mathrm{\acute{e}t}}) \cong \mathcal{P}_{\Sigma}(\mathrm{Fin})$ (the reference shows this holds before hypercompletion, and the right-hand side is hypercomplete by [BH17, Lemma 2.6]), i.e. the ∞ -category of product-preserving presheaves on finite sets, which is equivalent to the ∞ -category of anima, cf. [CS23, §5.1.4]. Unwinding the definitions, we see that this equivalence is precisely given by evaluation at k . \square

Lemma 2.37. *Suppose that k is a separably closed field. Then the functor*

$$(-)(k): \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_k) \rightarrow \mathcal{A}n$$

given by evaluation at k has a left and a right adjoint. In particular, it preserves all limits and colimits and commutes with p -completion, i.e. $L_p(F(k)) \cong (L_p F)(k)$ for all $F \in \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_k)$.

Proof. There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_k) & \xrightarrow{\iota^*} & \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(k_{\mathrm{\acute{e}t}}) \\ & \searrow (-)(k) & \cong \downarrow (-)(k) \\ & & \mathcal{A}n. \end{array}$$

Since k is separably closed, we have seen in Lemma 2.36 that the right vertical arrow is an equivalence. Moreover, ι^* has a left and a right adjoint, cf. Definition 2.27, hence the same is true for the diagonal functor. That evaluation at k then also commutes with p -completion follows formally from the fact that it preserves both p -equivalences and p -complete objects, cf. [Mat24b, Lemma 3.11]. \square

2.5 The canonical resolution of a connective sheaf

Let \mathcal{X} be an ∞ -topos. In this section, we prove that for every n -connective sheaf $X \in \mathcal{X}_{*, \geq n}$ there is a canonical way of writing X as a sifted colimit of sheaves of the form $\Sigma^n T$ with $T \in \mathcal{X}_*$.

Proposition 2.38. *Let $X \in \mathcal{X}_{*, \geq n}$. Then X is the geometric realization of the simplicial diagram given by $[k] \mapsto (\Sigma^n \Omega^n)^{k+1}(X)$. In particular, X is a sifted colimit of sheaves of the form $\Sigma^n T$ with $T \in \mathcal{X}_*$.*

Proof. If we can show that the functor $\Omega^n: \mathcal{X}_{*, \geq n} \rightarrow \mathcal{X}_*$ is monadic (i.e. is conservative and preserves geometric realizations of Ω^n -split simplicial diagrams), then this follows from (the proof of) [Lur17, Proposition 4.7.3.14]. Consider the following commutative diagram of right adjoints

$$\begin{array}{ccccc} \mathrm{Mon}_{\mathbb{E}_n}(\mathcal{X}_*) & \longleftarrow & \mathrm{Mon}_{\mathbb{E}_n}^{\mathrm{gp}}(\mathcal{X}_*) & \xleftarrow[\Omega^n]{\cong} & \mathcal{X}_{*, \geq n} \\ & \searrow \mathrm{fgt} & & \swarrow \Omega^n & \\ & & \mathcal{X}_* & & \end{array}$$

where the right horizontal morphism is an equivalence by [Lur17, Theorem 5.2.6.15]. Since $\mathrm{fgt}: \mathrm{Mon}_{\mathbb{E}_n}(\mathcal{X}_*) \rightarrow \mathcal{X}_*$ is monadic, it suffices to show that the forgetful functor $\mathrm{Mon}_{\mathbb{E}_n}^{\mathrm{gp}}(\mathcal{X}_*) \hookrightarrow \mathrm{Mon}_{\mathbb{E}_n}(\mathcal{X}_*)$ is conservative and preserves geometric realizations that are fgt-split. The first claim is clear. For the second claim, we have to show that a fgt-split geometric realization (in $\mathrm{Mon}_{\mathbb{E}_n}(\mathcal{X}_*)$) of grouplike \mathbb{E}_n -monoids is still grouplike. Grouplike objects are characterized by certain maps $G \times G \rightarrow G \times G$ being equivalences, cf. [Lur17, Definition 5.2.6.2]. Since geometric realizations that are fgt-split are computed underlying, and since colimits are universal in \mathcal{X} , the result follows. \square

3 \mathbb{A}^1 -invariance and étale sheaves

In this section, we will define \mathbb{A}^1 -invariant étale sheaves, and discuss (p -completely) étale \mathbb{A}^1 -nilpotent sheaves, a variant of nilpotence in an ∞ -topos, where the layers of the refined Postnikov tower are given by (p -completely) \mathbb{A}^1 -invariant infinite loop sheaves.

Definition 3.1. Let S be a scheme. Write $\mathrm{Spc}_{\mathrm{\acute{e}t}}(S) \subset \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)$ for the ∞ -category of \mathbb{A}^1 -invariant étale hypersheaves on Sm_S . As in the Nisnevich case, there is a Bousfield localization

$$L_{\mathrm{\acute{e}t}, \mathbb{A}^1} : \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S) \rightleftarrows \mathrm{Spc}_{\mathrm{\acute{e}t}}(S) : \iota_{\mathrm{\acute{e}t}, \mathbb{A}^1}$$

at the projections $X \times \mathbb{A}^1 \rightarrow X$ for $X \in \mathrm{Sm}_S$. We call objects of this ∞ -category *étale motivic spaces* over S . We write $\mathcal{SH}_{\mathrm{\acute{e}t}}^{S^1}(S) := \mathrm{Sp}(\mathrm{Spc}_{\mathrm{\acute{e}t}}(S))$ for the stabilization.

Recall the following lemma:

Lemma 3.2 (e.g. [Mat24b, Lemma A.1]). *Let S be a scheme. There is an adjunction*

$$L_{\mathrm{\acute{e}t}, \mathbb{A}^1} : \mathrm{Sp}(\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)) \rightleftarrows \mathcal{SH}_{\mathrm{\acute{e}t}}^{S^1}(S) : \iota_{\mathrm{\acute{e}t}, \mathbb{A}^1},$$

such that the following two diagrams are commutative:

$$\begin{array}{ccc} \mathrm{Sp}(\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)) & \xrightarrow{L_{\mathrm{\acute{e}t}, \mathbb{A}^1}} & \mathcal{SH}_{\mathrm{\acute{e}t}}^{S^1}(S) \\ \Sigma^\infty \uparrow & & \Sigma^\infty \uparrow \\ \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)_* & \xrightarrow{L_{\mathrm{\acute{e}t}, \mathbb{A}^1}} & \mathrm{Spc}_{\mathrm{\acute{e}t}}(S)_* \end{array} \quad \begin{array}{ccc} \mathrm{Sp}(\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)) & \xleftarrow{\iota_{\mathrm{\acute{e}t}, \mathbb{A}^1}} & \mathcal{SH}_{\mathrm{\acute{e}t}}^{S^1}(S) \\ \downarrow \Omega_*^\infty & & \downarrow \Omega_*^\infty \\ \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)_* & \xleftarrow{\iota_{\mathrm{\acute{e}t}, \mathbb{A}^1}} & \mathrm{Spc}_{\mathrm{\acute{e}t}}(S)_*. \end{array}$$

Lemma 3.3. *Let S be a scheme. The functor $\iota_{\mathrm{\acute{e}t}, \mathbb{A}^1} : \mathcal{SH}_{\mathrm{\acute{e}t}}^{S^1}(S) \rightarrow \mathrm{Sp}(\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S))$ is fully faithful, with essential image those étale hypersheaves of spectra that are \mathbb{A}^1 -invariant. In particular, $\mathcal{SH}_{\mathrm{\acute{e}t}}^{S^1}(S)$ is equivalent to the ∞ -category with the same name from [Bac21a, BH21].*

Proof. The functor is fully faithful by [Mat24b, Lemma A.2]. Everything in the essential image is \mathbb{A}^1 -invariant. Indeed, this can be checked after applying the functors $\Omega_*^\infty \Sigma^n$ for varying n , but now note that $\Omega_*^\infty \iota_{\mathrm{\acute{e}t}, \mathbb{A}^1} \cong \iota_{\mathrm{\acute{e}t}, \mathbb{A}^1} \Omega_*^\infty$ by Lemma 3.2. On the other hand, suppose that $E \in \mathrm{Sp}(\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S))$ is \mathbb{A}^1 -invariant. In particular, the $\Omega_*^\infty \Sigma^n E$ are all \mathbb{A}^1 -invariant and thus define an object of $\mathcal{SH}_{\mathrm{\acute{e}t}}^{S^1}(S)$. \square

Lemma 3.4. *Let \mathcal{X} be a Postnikov-complete ∞ -topos, and $T \in \mathcal{X}$ a coherent object of cohomological dimension $\leq N$ for some N . Then $T \in \mathcal{X}$ and $\Sigma_+^\infty T \in \mathrm{Sp}(\mathcal{X})$ are both compact.*

Proof. The second claim follows from the first one since the right adjoint Ω_*^∞ of Σ_+^∞ preserves filtered colimits (recall that in any ∞ -topos, filtered colimits commute with finite limits).

Now, $\tau_{\leq n} T$ is compact in $\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)_{\leq n}$ for every n by [Lur18, Corollary A.2.3.2]. Let X_i be a filtered system in \mathcal{X} . For every $n \geq 0$ we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Map}_{\mathcal{X}}(T, \mathrm{colim}_i X_i) & \xrightarrow{\quad\quad\quad} & \mathrm{colim}_i \mathrm{Map}_{\mathcal{X}}(T, X_i) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathcal{X}}(T, \tau_{\leq N+m} \mathrm{colim}_i X_i) & \xrightarrow{\quad\quad\quad} & \mathrm{colim}_i \mathrm{Map}_{\mathcal{X}}(T, \tau_{\leq N+m} X_i) \\
\downarrow \cong & & \downarrow \cong \\
\mathrm{Map}_{\mathcal{X}_{\leq N+m}}(\tau_{\leq N+m} T, \tau_{\leq N+m} \mathrm{colim}_i X_i) & \xrightarrow{\cong} & \mathrm{colim}_i \mathrm{Map}_{\mathcal{X}_{\leq N+m}}(\tau_{\leq N+m} T, \tau_{\leq N+m} X_i)
\end{array}$$

Here, the bottom vertical maps are equivalences by adjunction, and the bottom horizontal map is an equivalence since $\tau_{\leq N+m} T$ is compact in $\mathcal{X}_{\leq N+m}$, and $\tau_{\leq N+m} : \mathcal{X} \rightarrow \mathcal{X}_{\leq N+m}$ preserves colimits. Hence, also the middle horizontal map is an equivalence. By the proof of [Lur18, Proposition 1.3.3.10], using Postnikov-completeness of \mathcal{X} , the object T has homotopy dimension $\leq N$. Hence, both top vertical maps are $(m+1)$ -connective by Lemma 2.32. Hence, also the top horizontal map is $(m+1)$ -connective. Since m was arbitrary, we conclude that the top horizontal map is ∞ -connective, hence an equivalence of anima. \square

Lemma 3.5. *Let S be a qcqs scheme of finite Krull-dimension, that is moreover étale bounded. The functor $\iota_{\mathrm{ét}, \mathbb{A}^1} : \mathrm{Spc}_{\mathrm{ét}}(S) \rightarrow \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S)$ commutes with filtered colimits, and the functor $\iota_{\mathrm{ét}, \mathbb{A}^1} : \mathcal{SH}_{\mathrm{ét}}^{S^1}(S) \rightarrow \mathrm{Sp}(\mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S))$ commutes with all colimits.*

Proof. Since $\iota_{\mathrm{ét}, \mathbb{A}^1} : \mathcal{SH}_{\mathrm{ét}}^{S^1}(S) \rightarrow \mathrm{Sp}(\mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S))$ is exact, it commutes with finite colimits. Therefore, we only have to show that it commutes with filtered colimits.

As \mathbb{A}^1 -invariant objects are the local objects with respect to the morphisms $T \times \mathbb{A}^1 \rightarrow T$ (resp. $\Sigma_+^\infty(T \times \mathbb{A}^1) \rightarrow \Sigma_+^\infty T$), it suffices to show that T (resp. $\Sigma_+^\infty T$) is compact for every $T \in \mathrm{Sm}_S$.

Any $T \in \mathrm{Sm}_S$ has finite cohomological dimension by Lemmas 2.21 and 2.24. As the étale topology is finitary, we see from [Lur18, Proposition A.3.1.3] that $T \in \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S)$ is coherent. Hence, both T and $\Sigma_+^\infty T$ are compact by Lemma 3.4, which immediately implies the lemma. \square

Proposition 3.6. *Let S be a qcqs scheme of finite Krull-dimension, that is moreover étale bounded. The \mathbb{A}^1 -localization functor*

$$L_{\mathrm{ét}, \mathbb{A}^1} : \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S) \rightarrow \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S)$$

preserves connected objects.

Proof. As the étale topology is finitary, étale hypersheaves are stable under filtered colimits in the ∞ -category of presheaves. Hence, the same proof as for the Nisnevich topology applies (but with Nisnevich sheafification replaced by étale hypersheafification), see e.g. [Bac24, Lemma 1.2]. \square

Proposition 3.7. *Let S be a qcqs scheme of finite Krull-dimension, that is moreover étale bounded. Let $X \in \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S)_*$. If X is \mathbb{A}^1 -invariant, then also $\tau_{\geq 1} X$ is \mathbb{A}^1 -invariant.*

Proof. Consider the counit $\tau_{\geq 1} X \rightarrow X$. Since X is \mathbb{A}^1 -invariant by assumption, this map factors over the canonical map $\tau_{\geq 1} X \rightarrow L_{\mathrm{ét}, \mathbb{A}^1} \tau_{\geq 1} X$. As $L_{\mathrm{ét}, \mathbb{A}^1} \tau_{\geq 1} X$ is connected by Proposition 3.6, it follows that $L_{\mathrm{ét}, \mathbb{A}^1} \tau_{\geq 1} X \rightarrow X$ factors over $\tau_{\geq 1} X \rightarrow X$. Hence, we have the following commutative diagram:

$$\begin{array}{ccccc}
\tau_{\geq 1} X & \longrightarrow & L_{\text{ét}, \mathbb{A}^1} \tau_{\geq 1} X & \longrightarrow & \tau_{\geq 1} X \\
& & \downarrow & & \swarrow \\
& & X & &
\end{array}$$

where the top composition is the identity on $\tau_{\geq 1} X$. As \mathbb{A}^1 -invariant objects are stable under retracts (this holds for the local objects of any Bousfield localization), the proposition follows. \square

Lemma 3.8. *Let S be a qcqs scheme of finite Krull dimension, that is moreover étale bounded, and $\bar{s} \in S$ a geometric point. The functor*

$$\rho_{\bar{s}}^*: \text{Shv}_{\text{ét}}^h(\text{Sm}_S) \rightarrow \text{Shv}_{\text{ét}}^h(\text{Sm}_{S_{\bar{s}}^{\text{sh}}})$$

from Definition 2.30 preserves \mathbb{A}^1 -invariant sheaves.

Proof. Let $X \in \text{Shv}_{\text{ét}}^h(\text{Sm}_S)$ be \mathbb{A}^1 -invariant. By definition, the strict henselization

$$S_{\bar{s}}^{\text{sh}} \cong \lim_{\bar{s} \rightarrow U \xrightarrow{\text{ét}} S} U$$

is the limit over all étale neighborhoods U of \bar{s} in S . Let $T \in \text{Sm}_{S_{\bar{s}}^{\text{sh}}}$, so that by Lemma 2.31 we may choose a cofinal system of étale neighborhoods $\bar{s} \rightarrow U_i \rightarrow S$ and for every i a smooth U_i -scheme T_i such that $T \cong \lim_i T_i$. By Lemma 2.33 we have that $\rho_{\bar{s}}^*(X)(T) \cong \text{colim}_i X(T_i)$. The result follows from \mathbb{A}^1 -invariance of X and the fact that $\lim_i (T_i \times \mathbb{A}^1) \cong T \times \mathbb{A}^1$. \square

Lemma 3.9. *Let S be a scheme of finite Krull dimension that is moreover étale bounded. The collection of functors*

$$\rho_{\bar{s}}^*: \text{Shv}_{\text{ét}}^h(\text{Sm}_S) \rightarrow \text{Shv}_{\text{ét}}^h(\text{Sm}_{S_{\bar{s}}^{\text{sh}}})$$

from Definition 2.30 where \bar{s} ranges over the collection of geometric points of S jointly detect \mathbb{A}^1 -invariant objects, i.e. if $\rho_{\bar{s}}^* X$ is \mathbb{A}^1 -invariant for all \bar{s} , then X is \mathbb{A}^1 -invariant.

Proof. Let $X \in \text{Shv}_{\text{ét}}^h(\text{Sm}_S)$, such that $\rho_{\bar{s}}^* X$ is \mathbb{A}^1 -invariant for all \bar{s} . Consider the canonical map $f: X \rightarrow L_{\text{ét}, \mathbb{A}^1} X$. We have seen in Lemma 3.8 that $\rho_{\bar{s}}^* L_{\text{ét}, \mathbb{A}^1} X$ is \mathbb{A}^1 -invariant. Moreover, since $\rho_{\bar{s}}^*(\mathbb{A}_S^1 \times_S T) = \mathbb{A}_{S_{\bar{s}}^{\text{sh}}}^1 \times_{S_{\bar{s}}^{\text{sh}}} \rho_{\bar{s}}^* T$ for every smooth S -scheme T , it follows immediately that $\rho_{\bar{s}}^*$ preserves \mathbb{A}^1 -equivalences. In particular, we see that $\rho_{\bar{s}}^* f$ is an \mathbb{A}^1 -equivalence between \mathbb{A}^1 -invariant sheaves, hence an equivalence. This allows us to conclude from Lemma 2.35 that already f is an equivalence, i.e. X is \mathbb{A}^1 -invariant. \square

Definition 3.10. Let S be a scheme. Let $\mathcal{L}: \text{Sp}(\text{Shv}_{\text{ét}}^h(\text{Sm}_S)) \rightarrow \text{Sp}(\text{Shv}_{\text{ét}}^h(\text{Sm}_S))$ be a left Bousfield localization. Let $X \in \text{Shv}_{\text{ét}}^h(\text{Sm}_S)_*$. We say that X is \mathcal{L} -locally étale \mathbb{A}^1 -nilpotent if there exists a highly connected tower $(X_n)_n$ under X such that the following holds:

1. $X_0 \cong *$.
2. $X \cong \lim_n X_n$.

3. Each of the morphisms $X_{n+1} \rightarrow X_n$ is part of a fiber sequence $X_{n+1} \rightarrow X_n \rightarrow K_n$ where $K_n \cong \Omega_*^\infty E_n$ is a connected infinite loop sheaf for some $E_n \in \mathrm{Sp}(\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S))_{\geq 1}$.
4. For every n the object $\mathcal{L}E_n$ is \mathbb{A}^1 -invariant.

In the sequel, we will consider the following three special cases:

- If $\mathcal{L} = \mathrm{id}$ is the identity functor, we will say that X is *étale \mathbb{A}^1 -nilpotent*.
- If $\mathcal{L} = L_p$ is p -completion for some prime p , we will say that X is *p -completely étale \mathbb{A}^1 -nilpotent*.
- If $\mathcal{L} = L_{\mathbb{Q}}$ is rationalization, we will say that X is *rationally étale \mathbb{A}^1 -nilpotent*.

Lemma 3.11. *Let S be a scheme. Let $X \in \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)_*$ be étale \mathbb{A}^1 -nilpotent. Then X is p -completely étale \mathbb{A}^1 -nilpotent for every prime p , and moreover rationally étale \mathbb{A}^1 -nilpotent.*

Proof. Let $E \in \mathrm{Sp}(\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S))$ be \mathbb{A}^1 -invariant. Then $L_p E \cong \lim_n E//p^n$ by [Mat24b, Lemma 2.5], and hence is \mathbb{A}^1 -invariant as a limit of \mathbb{A}^1 -invariant sheaves. Similarly, $L_{\mathbb{Q}} E$ can be written as a filtered colimit of copies of E by [Mat24a, Lemma 2.6]. Hence, it is \mathbb{A}^1 -invariant since \mathbb{A}^1 -invariant sheaves of spectra are stable under colimits by Lemma 3.5. These two observations immediately imply the lemma. \square

Lemma 3.12. *Let S be a scheme. Let $X \in \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)_*$ be étale \mathbb{A}^1 -nilpotent (resp. p -completely étale \mathbb{A}^1 -nilpotent, resp. rationally étale \mathbb{A}^1 -nilpotent), with sheaves (X_n, K_n, E_n) as in Definition 3.10. Then K_n (resp. $L_p K_n$, resp. $L_{\mathbb{Q}} K_n$) is \mathbb{A}^1 -invariant.*

Proof. If X is étale \mathbb{A}^1 -nilpotent, then E_n is \mathbb{A}^1 -invariant, and hence so is $K_n \cong \Omega_*^\infty E_n$ by Lemma 3.2.

If X is p -completely \mathbb{A}^1 -nilpotent, then $L_p E_n$ is \mathbb{A}^1 -invariant. We have $L_p K_n \cong L_p \Omega_*^\infty E_n \cong \tau_{\geq 1} \Omega_*^\infty L_p E_n$ by [Mat24b, Lemma 3.17] (using that E_n is 1-connective). Hence, $L_p K_n$ is \mathbb{A}^1 -invariant by Lemma 3.2 and Proposition 3.7.

Similarly, if X is rationally \mathbb{A}^1 -nilpotent, then $L_{\mathbb{Q}} E_n$ is \mathbb{A}^1 -invariant. We have $L_{\mathbb{Q}} K_n \cong L_{\mathbb{Q}} \Omega_*^\infty E_n \cong \Omega_*^\infty L_{\mathbb{Q}} E_n$ by [Mat24a, Lemma 3.18] (using that E_n is 1-connective). Hence, $L_{\mathbb{Q}} K_n$ is \mathbb{A}^1 -invariant by Lemma 3.2. \square

Lemma 3.13. *Let S be a qcqs scheme of finite Krull-dimension, that is moreover étale bounded. Let $X \in \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)_*$ be étale \mathbb{A}^1 -nilpotent (resp. p -completely étale \mathbb{A}^1 -nilpotent, resp. rationally étale \mathbb{A}^1 -nilpotent). Then X (resp. $L_p X$, resp. $L_{\mathbb{Q}} X$) is \mathbb{A}^1 -invariant.*

Proof. Let \mathcal{L} be one of id , L_p or $L_{\mathbb{Q}}$, and suppose that X is \mathcal{L} -locally étale \mathbb{A}^1 -nilpotent. Choose sheaves $(X_n)_n$, $(K_n)_n$ and sheaves of spectra $(E_n)_n$ as in Definition 3.10. All the sheaves X_n are nilpotent: As $X_0 = *$ is nilpotent, this follows inductively, using the fiber sequence $X_{n+1} \rightarrow X_n \rightarrow K_n$ and [Mat24b, Lemma A.12], and the fact that K_n as a connected infinite loop sheaf is also nilpotent, cf. [Mat24b, Lemma A.11].

Note that $\mathcal{L}X \cong \lim_n \mathcal{L}X_n$: for $\mathcal{L} = \mathrm{id}$ this is by definition, for $\mathcal{L} = L_p$ we use Proposition 2.13, whereas for $\mathcal{L} = L_{\mathbb{Q}}$ we use [Mat24a, Proposition 6.9] (for

the latter two cases we need that $\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)$ admits a locally finite dimensional cover, which was shown in Proposition 2.25). As \mathbb{A}^1 -invariant sheaves are stable under limits, it is sufficient to inductively show that $\mathcal{L}X_n$ is \mathbb{A}^1 -invariant. For $n = 0$ this is clear as $\mathcal{L}X_0 = \mathcal{L}* \cong *$. For $n \geq 0$ consider the fiber sequence $X_{n+1} \rightarrow X_n \rightarrow K_n$. Applying \mathcal{L} we get

$$\mathcal{L}X_{n+1} \cong \tau_{\geq 1} \mathrm{fib}(\mathcal{L}X_n \rightarrow \mathcal{L}K_n),$$

this is clear if $\mathcal{L} = \mathrm{id}$, and follows from Proposition 2.8 in the case of $\mathcal{L} = L_p$, and from [Mat24a, Lemma 3.13] in the case of $\mathcal{L} = L_{\mathbb{Q}}$. As $\mathcal{L}X_n$ is \mathbb{A}^1 -invariant by the induction hypothesis, and $\mathcal{L}K_n$ is \mathbb{A}^1 -invariant by Lemma 3.12, we get that also $\mathcal{L}X_{n+1}$ is \mathbb{A}^1 -invariant (using again Proposition 3.7). \square

4 Rational unstable étale motives

In this section we prove that over a perfect field of finite cohomological dimension any nilpotent and 2-effective motivic space satisfies étale hyperdescent.

Proposition 4.1. *Let k be a perfect field, with $\mathrm{cd}(k) < \infty$. Let $E \in \mathcal{SH}(k)$ be rational. Then the Nisnevich sheaf of spectra $\omega^\infty E \in \mathcal{SH}^{S^1}(k)$ satisfies étale hyperdescent.*

Proof. First note that there is an equivalence $\mathcal{SH}(S)_{\mathbb{Q}} \cong \mathcal{D}_{\mathbb{A}^1}(S, \mathbb{Q})$ between the ∞ -category of rational motivic spectra and Morel's \mathbb{A}^1 -derived category with rational coefficients, cf. [CD19, §5.3.35]; we will use this in the following without mention. Note that there is a decomposition $E \cong E_+ \oplus E_-$ by [CD19, §16.2.1]. It follows from [CD19, Corollary 16.2.14] that $E_- \cong 0$ (as $\mathrm{cd}(k) < \infty$), and thus from [CD19, Theorem 16.2.18] that $\omega^\infty E \cong \omega^\infty E_+$ satisfies étale hyperdescent (note that in the reference they use the word *descent* for the word *hyperdescent*). \square

Theorem 4.2. *Let k be a perfect field, with $\mathrm{cd}(k) < \infty$. Let $X \in \mathrm{Spc}(k)_*$ be nilpotent, 2-effective and rational. Then X satisfies étale hyperdescent.*

Proof. We use the version of the Postnikov-tower from Proposition 2.14, i.e. we have a tower of pointed motivic spaces $X_n \in \mathrm{Spc}(k)_*$ under X with $X_0 = *$, and 2-effective motivic spectra $E_n \in \mathcal{SH}(k)$, together with fiber sequences $X_{n+1} \rightarrow X_n \rightarrow \Omega_{\mathbb{P}^1, *}^\infty E_n$, and an equivalence $X \cong \lim_n X_n$. By Lemma 2.16 we may further assume that all the X_n and E_n are rational. Thus, as étale hyper-sheaves are stable under limits, by induction it suffices to show that $\Omega_{\mathbb{P}^1, *}^\infty E_n$ satisfies étale hyperdescent for all n . This follows, as already $\omega^\infty E_n$ satisfies étale hyperdescent, cf. Proposition 4.1. \square

5 Unstable étale motives at the characteristic

In this section, we prove some vanishing results for \mathbb{A}^1 -invariant sheaves at the characteristic.

Proposition 5.1. *Let p be a prime and S be an \mathbb{F}_p -scheme of finite Krull dimension which is moreover étale bounded. If $X \in \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)$ is p -completely étale \mathbb{A}^1 -nilpotent, then $L_p X \cong *$.*

Proof. By definition of étale \mathbb{A}^1 -nilpotent there is a sequence of nilpotent sheaves X_n under X , with $X \cong \lim_n X_n$, and fiber sequences $X_{n+1} \rightarrow X_n \rightarrow K_n$. Moreover, $K_n \cong \Omega_*^\infty E_n$, where E_n is a 1-connective étale sheaf of spectra, such that $L_p E_n$ is \mathbb{A}^1 -invariant. It follows from [BH21, Theorem A.1] that $L_p E_n \cong 0$, and hence $L_p K_n \cong \tau_{\geq 1} \Omega_*^\infty L_p E_n \cong *$, where we used [Mat24b, Lemma 3.17]. Using Proposition 2.8 we see inductively

$$L_p X_{n+1} \cong \tau_{\geq 1} \mathrm{fib}(L_p X_n \rightarrow L_p K_n) \cong \tau_{\geq 1} \mathrm{fib}(* \rightarrow *) \cong *.$$

Hence, using Proposition 2.13, we also get $L_p X \cong \lim_n \tau_{\leq n} L_p X_n \cong *$, which proves the proposition. \square

Proposition 5.2. *Let p be a prime and k be a perfect field of characteristic p . Let $M \in \mathrm{Mod}_{H\mathbb{F}_p}(\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S, \mathrm{Sp}))_{\geq 2}$ be a 2-connective sheaf of $H\mathbb{F}_p$ -modules. Suppose that $\Omega_*^\infty M$ is \mathbb{A}^1 -invariant. Then $\Omega_*^\infty M \cong *$ and $M \cong 0$.*

Proof. Since M is connective, it is clear that $\Omega_*^\infty M \cong *$ implies $M \cong 0$. Hence, it suffices to prove the first claim. Consider the Artin–Schreier sequence

$$H\mathbb{F}_p \rightarrow H\mathbb{G}_a \xrightarrow{1-F} H\mathbb{G}_a$$

as a fiber sequence in $\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S, \mathrm{Sp})$. Here, we view $\mathbb{G}_a := \mathbb{A}^1$ as an étale sheaf of \mathbb{F}_p -vectorspaces. Tensoring with M over $H\mathbb{F}_p$ gives a fiber sequence

$$M \rightarrow M \otimes_{H\mathbb{F}_p} H\mathbb{G}_a \rightarrow M \otimes_{H\mathbb{F}_p} H\mathbb{G}_a,$$

and by rotating twice and applying Ω_*^∞ also a fiber sequence

$$\Omega_*^\infty(M \otimes_{H\mathbb{F}_p} H\mathbb{G}_a) \rightarrow \Omega_*^\infty(M \otimes_{H\mathbb{F}_p} H\mathbb{G}_a) \rightarrow \Omega_*^\infty M.$$

We claim that applying $L_{\mathbb{A}^1, \mathrm{nis}}$ preserves this fiber sequence, i.e. that we get a fiber sequence

$$L_{\mathbb{A}^1, \mathrm{nis}} \Omega_*^\infty(M \otimes_{H\mathbb{F}_p} H\mathbb{G}_a) \rightarrow L_{\mathbb{A}^1, \mathrm{nis}} \Omega_*^\infty(M \otimes_{H\mathbb{F}_p} H\mathbb{G}_a) \rightarrow L_{\mathbb{A}^1, \mathrm{nis}} \Omega_*^\infty M.$$

This follows from [AWW17, Theorem 2.3.3] if we can show that $\pi_0^{\mathrm{nis}}(\Omega_*^\infty M) \cong *$, and that $\pi_1^{\mathrm{nis}}(\Omega_*^\infty M)$ is strongly \mathbb{A}^1 -invariant. The second claim follows from a theorem of Morel [Bac24, Corollary 1.8] since $\Omega_*^\infty M$ is assumed to be \mathbb{A}^1 -invariant and is even an étale sheaf. For the first claim, note that $\pi_0^{\mathrm{nis}}(\Omega_*^\infty M) \cong \pi_0^{\mathrm{nis}}(M) \cong L_{\mathrm{nis}}(U \mapsto \pi_0(M(U)))$. By Zariski descent it therefore suffices to show that $\pi_0(M(U)) \cong 0$ for all affine $U \in \mathrm{Sm}_k$. There is a spectral sequence $E_2^{p,q} = H_{\mathrm{\acute{e}t}}^p(U, \pi_q(M)) \implies \pi_{q-p}(M(U))$. Since M is 2-connective by assumption, it therefore suffices to show that $H_{\mathrm{\acute{e}t}}^p(U, \pi_q(M)) = 0$ for all $p \geq 2$. This is an immediate application of [GAV71, Théorème X.5.1] since $\pi_q(M)$ is a sheaf of \mathbb{F}_p -vectorspaces by assumption and U was assumed to be affine.

Now, since $\Omega_*^\infty M$ is assumed to be \mathbb{A}^1 -invariant, and is moreover even an étale sheaf, we see that $\Omega_*^\infty M \cong L_{\mathbb{A}^1, \mathrm{nis}} \Omega_*^\infty M$. We claim that it suffices to show that $L_{\mathbb{A}^1, \mathrm{nis}} \Omega_*^\infty(M \otimes_{H\mathbb{F}_p} H\mathbb{G}_a) \cong *$. Indeed, we have seen above that $\Omega_*^\infty M$ is connected as a Nisnevich sheaf, and hence the long exact sequence

in Nisnevich homotopy sheaves shows that $\Omega_*^\infty M \cong *$. Since \mathbb{G}_a admits the structure of a sheaf of \mathbb{F}_p -algebras, we see that $H\mathbb{G}_a$ is an \mathbb{E}_∞ -algebra in $\text{Mod}_{H\mathbb{F}_p}(\text{Shv}_{\text{ét}}^h(\text{Sm}_S, \text{Sp}))$. In particular, $\Omega M \otimes_{H\mathbb{F}_p} H\mathbb{G}_a$ is canonically an $H\mathbb{G}_a$ -module. Since Ω_*^∞ and $L_{\mathbb{A}^1, \text{nis}}$ are both lax symmetric monoidal, we see that $L_{\mathbb{A}^1, \text{nis}}\Omega_*^\infty(\Omega M \otimes_{H\mathbb{F}_p} H\mathbb{G}_a)$ acquires the structure of a module over the commutative monoid $L_{\mathbb{A}^1, \text{nis}}\Omega_*^\infty(H\mathbb{G}_a) \cong L_{\mathbb{A}^1, \text{nis}}\mathbb{G}_a \cong *$ (using that the underlying sheaf of \mathbb{G}_a is just \mathbb{A}^1). Hence, $L_{\mathbb{A}^1, \text{nis}}\Omega_*^\infty(\Omega M \otimes_{H\mathbb{F}_p} H\mathbb{G}_a) \cong *$. \square

Corollary 5.3. *Let p be a prime and k be a perfect field of characteristic p . Let $M \in \text{Mod}_{H\mathbb{Z}}(\text{Shv}_{\text{ét}}^h(\text{Sm}_S, \text{Sp}))_{\geq 3}$ be a 3-connective sheaf of $H\mathbb{Z}$ -modules. Suppose that $L_p\Omega_*^\infty M$ is \mathbb{A}^1 -invariant. Then $L_p\Omega_*^\infty M \cong *$ and $L_p M \cong 0$.*

Proof. Note that $M//p$ is an $H\mathbb{F}_p$ -module (since we assume M to be an $H\mathbb{Z}$ -module). There is a fiber sequence $\Omega M//p \rightarrow M \rightarrow M$, which is preserved by Ω_*^∞ . Applying Proposition 2.8 we thus get an equivalence

$$L_p\Omega_*^\infty \Omega M//p \cong \tau_{\geq 1} \text{fib}(L_p\Omega_*^\infty M \rightarrow L_p\Omega_*^\infty M).$$

Since $M//p$ is an $H\mathbb{F}_p$ -module, we see that $\Omega_*^\infty \Omega M//p$ is already p -complete. Hence,

$$\Omega_*^\infty \Omega M//p \cong \tau_{\geq 1} \text{fib}(L_p\Omega_*^\infty M \rightarrow L_p\Omega_*^\infty M).$$

Since $L_p\Omega_*^\infty M$ is assumed to be \mathbb{A}^1 -invariant, and \mathbb{A}^1 -invariant sheaves are stable under limits and connected covers (see Proposition 3.7 for the latter), we conclude that $\Omega_*^\infty \Omega M//p$ is \mathbb{A}^1 -invariant. Therefore, it follows from Proposition 5.2 that $M//p \cong 0$, and hence $L_p M \cong 0$. In particular, also $L_p\Omega_*^\infty M \cong \tau_{\geq 1}\Omega_*^\infty L_p M \cong *$, where we used [Mat24b, Corollary 3.18]. \square

Corollary 5.4. *Let p be a prime and k be a perfect field of characteristic p . Suppose that $A \in \text{Shv}_{\text{ét}}(\text{Sm}_k, \mathcal{A}b)$ is a sheaf of abelian groups and $i \geq 3$ such that $L_p K(A, i)$ is \mathbb{A}^1 -invariant. Then $L_p K(A, n) \cong *$ for all $n \geq 1$ and $L_p H A \cong 0$.*

Proof. Note that $\Sigma^k H A$ is in the (shift of the) heart, and hence admits an $H\mathbb{Z}$ -module structure. Moreover, $K(A, i) \cong \Omega_*^\infty \Sigma^k H A$. Hence, we see from Corollary 5.3 that $L_p H A \cong 0$. Now, for any $n \geq 1$ we have

$$L_p K(A, n) \cong \tau_{\geq 1} \Omega_*^\infty \Sigma^n L_p H A \cong *,$$

using [Mat24b, Corollary 3.18]. \square

Remark 5.5. Note that the Proposition 5.2 does not immediately follow from [BH21, Theorem A.1], as it is not clear that the \mathbb{A}^1 -invariance of $\Omega_*^\infty M$ implies the \mathbb{A}^1 -invariance of M , not even in the case that M is 2-connective (*a posteriori* it does as $M \cong 0$ in this case). A similar remark applies to Corollaries 5.3 and 5.4.

6 Proof of the rigidity theorem

In this section we prove our main result, the unstable rigidity theorem. Let S be a qcqs scheme with finite Krull-dimension, such that S is étale bounded. Moreover, let p be a prime invertible on S .

Lemma 6.1. *The functor $\iota_! : \mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{h}}(S_{\acute{\mathrm{e}}\mathrm{t}})_* \rightarrow \mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{h}}(\mathrm{Sm}_S)_*$ from Definition 2.27 preserves nilpotent objects.*

Proof. The statement is true for any geometric morphism, since they commute with taking homotopy objects [Lur09, Remark 6.5.1.4]. \square

Lemma 6.2. *The adjunctions $\iota_! \dashv \iota^* \dashv \iota_*$ induce adjunctions*

$$\iota_!^p : \mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{h}}(S_{\acute{\mathrm{e}}\mathrm{t}})_{*,p}^{\wedge} \rightleftarrows \mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{h}}(\mathrm{Sm}_S)_{*,p}^{\wedge} : \iota_p^*$$

and

$$\iota_p^* : \mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{h}}(\mathrm{Sm}_S)_{*,p}^{\wedge} \rightleftarrows \mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{h}}(S_{\acute{\mathrm{e}}\mathrm{t}})_{*,p}^{\wedge} : \iota_*^p.$$

Here, ι_p^* and ι_*^p are just given by restriction of ι^* and ι_* , respectively.

Proof. Note that ι^* restricts to a functor $\iota_p^* : \mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{h}}(\mathrm{Sm}_S)_{*,p}^{\wedge} \rightarrow \mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{h}}(S_{\acute{\mathrm{e}}\mathrm{t}})_{*,p}^{\wedge}$, since it preserves p -complete objects as a right adjoint (this follows formally from [Mat24b, Lemma 3.11]). Similarly, ι_* restricts to a functor $\iota_*^p : \mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{h}}(S_{\acute{\mathrm{e}}\mathrm{t}})_{*,p}^{\wedge} \rightarrow \mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{h}}(\mathrm{Sm}_S)_{*,p}^{\wedge}$. Write $\iota_!^p$ for the composition $L_p \circ \iota_!|_{\mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{h}}(S_{\acute{\mathrm{e}}\mathrm{t}})_{*,p}^{\wedge}}$. It is formal that this gives adjunctions $\iota_!^p \dashv \iota_p^* \dashv \iota_*^p$. \square

Lemma 6.3. *The functors $\iota_!^p$ and ι_*^p are fully faithful.*

Proof. It suffices to show that the double right adjoint ι_*^p is fully faithful (cf. (the dual of) [NPR24, Corollary 2.7]). This is clear since ι_*^p is just the restriction of the fully faithful functor ι_* , cf. Lemma 2.28. \square

Proposition 6.4. *Let $X \in \mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{h}}(S_{\acute{\mathrm{e}}\mathrm{t}})_*$ be nilpotent. Then $\iota_! X$ is p -completely étale \mathbb{A}^1 -nilpotent (and in particular $L_p \iota_! X$ is \mathbb{A}^1 -invariant).*

Proof. Since X is nilpotent, we can choose a principal refinement $(X_n, K_n)_n$ of the Postnikov-tower of X with $K_n \cong \Omega_{*}^{\infty} E_n$ for some 2-connective sheaf of spectra E_n [Mat24b, Lemma A.15]. Then $(\iota_! X_n, \iota_! K_n)_n$ is a refinement of the Postnikov tower of $\iota_! X$. Indeed, it is clear that this gives a refinement of the Postnikov tower of $\lim_n \iota_! X_n$. Moreover, $\iota_!$ commutes with this limit by Proposition 2.1, as $\mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{h}}(S_{\acute{\mathrm{e}}\mathrm{t}})$ is locally of homotopy dimension $\leq N$ for some N , cf. Lemma 2.23, and $\mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{h}}(\mathrm{Sm}_S)$ has a locally finite dimensional cover, cf. Proposition 2.25. Note that $\iota_! K_n \cong \Omega_{*}^{\infty} \iota_! E_n$. It therefore suffices to show that $L_p \iota_! E_n$ is \mathbb{A}^1 -invariant, which follows from the proof of [Bac21a, Corollary 6.2], since $L_p \iota_! E_n \cong L_p \iota_! L_p E_n$, as ι^* preserve p -equivalences.

That $L_p \iota_! X$ is \mathbb{A}^1 -invariant now follows from Lemma 3.13. \square

Definition 6.5. Let $X \in \mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{h}}(\mathrm{Sm}_S)_*$. We say that X is p -completely small if the counit $\iota_! \iota^* X \rightarrow X$ is a p -equivalence, i.e. if and only if $L_p \iota_! \iota^* X \rightarrow L_p X$ is an equivalence. In particular, in this case $L_p X$ is in the essential image of $\iota_!^p$.

Theorem 6.6 (Essential Image). *Let $W \in \mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{h}}(\mathrm{Sm}_S)_*$ be p -completely étale \mathbb{A}^1 -nilpotent. Then W is p -completely small. In particular, $L_p W$ is in the essential image of $\iota_!^p$, restricted to the subcategory of p -complete nilpotent objects.*

Proof. The “in particular” part follows as both $\iota_!$ and ι^* preserve p -equivalences (since they are left adjoints), and hence $L_p \iota_! \iota^* \cong \iota_!^p \iota_p^* L_p$. By assumption there is an \mathbb{N} -indexed system $(W_n)_n$ under W with $\lim_n W_n \cong W$, $W_0 \cong *$ and fiber

sequences $W_{n+1} \rightarrow W_n \rightarrow K_n$ for some sheaves K_n such that $K_n \cong \Omega_*^\infty E_n$, where $E_n \in \mathrm{Sp}(\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S))_{\geq 1}$ is a 1-connective sheaf of spectra, such that $L_p E_n$ is \mathbb{A}^1 -invariant. We proceed in several steps:

Step 1: We prove that $L_p K_n \cong L_p \iota_! \iota^* K_n$. Since E_n is 1-connective, it follows from [Mat24b, Lemma 3.17] that $L_p K_n \cong \tau_{\geq 1} \Omega_*^\infty L_p E_n$. Moreover, it follows from [Bac21a, Theorem 6.6] that $L_p E_n \cong L_p \iota_! \iota^* E_n$ (this is the stable rigidity theorem, which needs our assumptions on S . In the reference it is shown that $\iota_!$ induces an equivalence on p -complete objects, and hence the above claim follows since both $\iota_!$ and its right adjoint ι^* preserve p -equivalences). Therefore, we get

$$\begin{aligned} L_p K_n &\cong \tau_{\geq 1} \Omega_*^\infty L_p E_n \\ &\cong \tau_{\geq 1} \Omega_*^\infty L_p \iota_! \iota^* E_n \\ &\cong L_p \Omega_*^\infty \iota_! \iota^* E_n \\ &\cong L_p \iota_! \iota^* \Omega_*^\infty E_n \\ &\cong L_p \iota_! \iota^* K_n, \end{aligned}$$

where we used again [Mat24b, Lemma 3.17] in the third equivalence (as also $\iota_! \iota^* E_n$ is 1-connective as both functors are t-exact, cf. [Lur18, Remark 1.3.2.8]), and e.g. [Mat24b, Lemmas A.1 and A.3] in the fourth equivalence.

Step 2: We prove that $L_p W_n \cong L_p \iota_! \iota^* W_n$. We prove this by induction on n , using that for $W_0 = *$ the statement is true since all involved functors preserve the terminal object. Suppose now that $L_p W_n \cong L_p \iota_! \iota^* W_n$. We have

$$\begin{aligned} L_p \iota_! \iota^* W_{n+1} &\cong L_p \iota_! \iota^* \mathrm{fib}(W_n \rightarrow K_n) \\ &\cong L_p \mathrm{fib}(\iota_! \iota^* W_n \rightarrow \iota_! \iota^* K_n) \\ &\cong \tau_{\geq 1} \mathrm{fib}(L_p \iota_! \iota^* W_n \rightarrow L_p \iota_! \iota^* K_n), \end{aligned}$$

using in the second equivalence that both $\iota_!$ and ι^* preserve finite limits, as both are (the left adjoints of) geometric morphisms. The last equivalence holds by Proposition 2.8, since $\mathrm{fib}(\iota_! \iota^* W_n \rightarrow \iota_! \iota^* K_n) \cong \iota_! \iota^* W_{n+1}$ is connected. Analogously, one shows $L_p W_{n+1} \cong \tau_{\geq 1} \mathrm{fib}(L_p W_n \rightarrow L_p K_n)$. Thus, we conclude using the induction hypothesis and the first step.

Step 3: We prove that $L_p W \cong L_p \iota_! \iota^* W$. We calculate

$$\begin{aligned} L_p \iota_! \iota^* W &\cong L_p \iota_! \iota^* \lim_n W_n \\ &\cong L_p \iota_! \lim_n \iota^* W_n \\ &\cong L_p \lim_n \iota_! \iota^* W_n \\ &\cong \lim_n L_p \iota_! \iota^* W_n \\ &\cong \lim_n L_p W_n \\ &\cong L_p W. \end{aligned}$$

The first equivalence holds by definition of W . The functor ι^* commutes with the limit since it is a right adjoint, whereas $\iota_!$ preserves the limit by Proposition 2.1, using that $(W_n)_n$ and hence also $(\iota^* W_n)_n$ are highly connected towers. The fourth and last equivalence are applications of Proposition 2.13, using that $\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)$ admits a locally finite-dimensional cover by Proposition 2.25. The fifth equivalence is given by the second step. \square

Combining the above results we get the unstable rigidity theorem.

Corollary 6.7 (Étale rigidity for motivic spaces). *The functor*

$$\iota_!^p : \mathrm{Shv}_{\mathrm{ét}}^h(S_{\mathrm{ét}})_{*,p,\mathrm{nil}}^{\wedge} \rightarrow \mathrm{Shv}_{\mathrm{ét}}^h(\mathrm{Sm}_S)_{*,p}^{\wedge}$$

is fully faithful, with essential image those p -complete \mathbb{A}^1 -invariant sheaves that are the p -completion of a p -completely étale \mathbb{A}^1 -nilpotent sheaf.

Proof. We have seen in Lemma 6.3 that the functor $\iota_!^p$ is fully faithful. Note that by Proposition 6.4 it factors through the proclaimed essential image. Hence, the corollary follows from Theorem 6.6. \square

7 A retract of projective space

In this section let k be an algebraically closed field and $p \neq \mathrm{char}(k)$ a prime number. We prove that in étale \mathbb{A}^1 -homotopy theory over k there is a retract diagram $L_p S^2 \rightarrow L_p \mathbb{P}^1 \rightarrow L_p S^2$. Choose a compatible system of primitive roots of unity $(\zeta_{p^n})_n$, i.e. a sequence with $\zeta_{p^n} \in k^\times$ a primitive p^n -th root of unity such that moreover $(\zeta_{p^{n+1}})^p = \zeta_{p^n}$ for all n . In particular, $(\zeta_{p^n})_n$ defines an element of $\mathbb{Z}_p(1)(k) = \varinjlim_n k^\times[p^n]$.

Recall that in [Bac21a, Section 3] Bachmann constructed (the desuspension of) a map $\sigma : L_p L_{\mathrm{ét},\mathbb{A}^1} \mathbb{P}^1 \rightarrow L_p S^2$, and its stabilization $\sigma_{st} : L_p L_{\mathrm{ét},\mathbb{A}^1} \Sigma^\infty \mathbb{P}^1 \rightarrow L_p S^2 \in \mathcal{SH}_{\mathrm{ét}}^1(k)$ (since k is algebraically closed, we can choose a trivialization of the twisting space, cf. [Bac21a, Theorem 3.6 (3)]).

Our first goal is the construction of the other map of the retract. For this, we will work in the Nisnevich topology. We write $L_{\mathrm{nis},\mathbb{A}^1} : \mathcal{P}(\mathrm{Sm}_S) \rightarrow \mathrm{Spc}(k)$ for the motivic localization functor, and

$$H_{\mathrm{nis}} : \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_k, \mathcal{A}b) \cong \mathrm{Sp}(\mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_k))^{\heartsuit} \rightarrow \mathrm{Sp}(\mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_k))$$

for the embedding of the heart. Similarly, for every $n \geq 1$ we write

$$K_{\mathrm{nis}}(-, n) : \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_k, \mathcal{A}b) \rightarrow \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_k)$$

for the Nisnevich-local Eilenberg-MacLane sheaves. We begin by constructing an element $\tau_0 \in \pi_2(L_p K_{\mathrm{nis}}(\mathbb{G}_m, 1))(k)$:

Construction 7.1. Write $A[p^n] := \ker(A \xrightarrow{p^n} A)$ for the p^n -torsion of a sheaf of abelian groups A . Using [Mat24b, Corollary 3.18 and Lemma 2.24], we have

$$\pi_2(L_p K_{\mathrm{nis}}(\mathbb{G}_m, 1)) \cong \pi_2(L_p \Sigma^1 H_{\mathrm{nis}} \mathbb{G}_m) \cong \varinjlim_n \mathbb{G}_m[p^n] = \varinjlim_n \mu_{p^n} = \mathbb{Z}_p(1),$$

and therefore

$$\pi_2(L_p K_{\mathrm{nis}}(\mathbb{G}_m, 1))(k) \cong \mathbb{Z}_p(1)(k) = \varinjlim_n \mu_{p^n}(k).$$

Under this equivalence, we let $\tau_0 \in \pi_2(L_p K_{\mathrm{nis}}(\mathbb{G}_m, 1))(k)$ be the element given by the system of roots of unity $(\zeta_{p^n})_n$.

Lemma 7.2. *There is a map $\psi : \pi_2 L_p L_{\mathrm{nis},\mathbb{A}^1} \mathbb{P}^1 \rightarrow \pi_2 L_p K_{\mathrm{nis}}(\mathbb{G}_m, 1)$ such that τ_0 is in the image of this map.*

Proof. We will show that there exists a Nisnevich sheaf $F \in \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_k)_*$ that fits into an exact sequence of Nisnevich sheaves of groups

$$\pi_2 L_p L_{\mathrm{nis}, \mathbb{A}^1} \mathbb{P}^1 \xrightarrow{\psi} \pi_2 L_p K_{\mathrm{nis}}(\mathbb{G}_m, 1) \xrightarrow{\phi} \pi_1 L_p F$$

such that $\phi(\tau_0) = 0$. Recall from the discussion before [Mor12, Theorem 6.29] that there is a short exact sequence of Nisnevich sheaves of groups

$$0 \rightarrow \mathbf{K}_2^{\mathrm{MW}} \rightarrow \pi_1(L_{\mathrm{nis}, \mathbb{A}^1} \mathbb{P}^1) \rightarrow \mathbb{G}_m \rightarrow 0.$$

Write $F := \mathrm{fib}(L_{\mathrm{nis}, \mathbb{A}^1} \mathbb{P}^1 \rightarrow K_{\mathrm{nis}}(\mathbb{G}_m, 1))$ for the fiber of the canonical map $L_{\mathrm{nis}, \mathbb{A}^1} \mathbb{P}^1 \rightarrow K_{\mathrm{nis}}(\pi_1(L_{\mathrm{nis}, \mathbb{A}^1} \mathbb{P}^1), 1) \rightarrow K_{\mathrm{nis}}(\mathbb{G}_m, 1)$ induced by the right map in the above short exact sequence. Note that F is connected as $\pi_1(L_{\mathrm{nis}, \mathbb{A}^1} \mathbb{P}^1) \rightarrow \mathbb{G}_m$ is surjective and $L_{\mathrm{nis}, \mathbb{A}^1} \mathbb{P}^1 \cong L_{\mathrm{nis}, \mathbb{A}^1} \Sigma \mathbb{G}_m$ is connected. As $\pi_2 K_{\mathrm{nis}}(\mathbb{G}_m, 1) = 0$, we see that $\pi_1(F) \cong \mathbf{K}_2^{\mathrm{MW}}$.

Since $L_{\mathrm{nis}, \mathbb{A}^1} \mathbb{P}^1$ and $K_{\mathrm{nis}}(\mathbb{G}_m, 1)$ are nilpotent by [AFH22, Remark 3.4.9] and [Mat24b, Lemma A.11], it follows from Proposition 2.8 that after p -completing, we still have

$$L_p F \cong \tau_{\geq 1} \mathrm{fib}(L_p L_{\mathrm{nis}, \mathbb{A}^1} \mathbb{P}^1 \rightarrow L_p K_{\mathrm{nis}}(\mathbb{G}_m, 1)).$$

Thus, the long exact sequence in homotopy yields the wanted exact sequence.

It is left to show that $\phi(\tau_0) = 0$. As $\phi(\tau_0) \in (\pi_1 L_p F)(k)$, it clearly suffices to show that this group is 0. We have

$$(\pi_1 L_p F)(k) \cong \pi_1 L_p(F(k)) \cong \mathbb{L}_0 \pi_1(F(k)) \cong \mathbb{L}_0((\pi_1 F)(k))$$

by Lemma 2.36 and [MP11, Theorem 11.1.2], here \mathbb{L}_0 denotes the zeroth derived p -completion functor on the category of groups as defined e.g. in [MP11, Chapter 10.4] (where it is called \mathbb{E}_p). Now note that $(\pi_1 F)(k) = \mathbf{K}_2^{\mathrm{MW}}(k)$ is uniquely p -divisible as k is algebraically closed, for this combine e.g. [Dég23, Example 2.3.11] with [BT06, Chapter I, Corollary 1.3]. This immediately implies that $\mathbb{L}_0(\pi_1 F)(k) \cong 0$, cf. [MP11, Proposition 10.4.7 (iii)], which is what we wanted to show. \square

Construction 7.3. Hence, we can choose an element $\tau \in (\pi_2 L_p L_{\mathrm{nis}, \mathbb{A}^1} \mathbb{P}^1)(k)$ such that $\psi(\tau) = \tau_0$. Therefore, τ defines a map $S^2 \rightarrow L_p L_{\mathrm{nis}, \mathbb{A}^1} \mathbb{P}^1$, and thus also a map

$$\tau: L_p S^2 \rightarrow L_p L_{\mathrm{nis}, \mathbb{A}^1} \mathbb{P}^1,$$

which we will also denote by τ .

Bachmann constructed in [Bac21a, Lemma 6.4] (the desuspension of) a map $\tau_{st}: L_p \Sigma^2 \mathbb{S} \rightarrow L_p L_{\mathrm{nis}, \mathbb{A}^1} \Sigma^\infty \mathbb{P}^1$ in $\mathcal{SH}^{S^1}(k)$. His construction depends on a variety of choices, but he then shows that independently of these choices, τ_{st} is an equivalence. We will show that our map τ stabilizes to one of these maps τ_{st} . For this, let us make the following definition.

Definition 7.4. Let $f: L_p \Sigma^2 \mathbb{S} \rightarrow L_p L_{\mathrm{nis}, \mathbb{A}^1} \Sigma^\infty \mathbb{P}^1 \in \mathcal{SH}^{S^1}(k)$ be a map, or equivalently by adjunction (and up to homotopy), $f \in \pi_2(L_p L_{\mathrm{nis}, \mathbb{A}^1} \Sigma^\infty \mathbb{P}^1)(k)$ be a homotopy class. We say that f is *of stable τ -type* if the image of f under the chain of maps

$$\pi_2(L_p L_{\mathrm{nis}, \mathbb{A}^1} \Sigma^\infty \mathbb{P}^1)(k) \rightarrow \pi_2((L_{\mathrm{nis}, \mathbb{A}^1} \Sigma^\infty \mathbb{P}^1) // p)(k) \rightarrow \pi_1(L_{\mathrm{nis}, \mathbb{A}^1} \Sigma^\infty \mathbb{P}^1)(k) \cong \mathbf{K}_1^{\mathrm{MW}}(k) \quad (2)$$

is the element $[\zeta_p]$ corresponding to our chosen primitive p -th root of unity. Here, the first map is the projection out of the limit, the second map is the boundary map of the cofiber sequence $L_{\text{nis}, \mathbb{A}^1} \Sigma^\infty \mathbb{P}^1 \xrightarrow{p} L_{\text{nis}, \mathbb{A}^1} \Sigma^\infty \mathbb{P}^1 \rightarrow (L_{\text{nis}, \mathbb{A}^1} \Sigma^\infty \mathbb{P}^1) // p$, and the last isomorphism is [Mor12, Corollary 25].

We chose this definition because of the following result.

Proposition 7.5. *We have that $L_p \Sigma^\infty \tau: L_p \mathbb{S}^2 \rightarrow L_p L_{\text{nis}, \mathbb{A}^1} \Sigma^\infty \mathbb{P}^1 \in \mathcal{SH}^{S^1}(k)$ is of stable τ -type.*

Proof. By adjunction, $L_p \Sigma^\infty \tau$ corresponds to a $\tilde{\tau} \in \pi_2(L_p L_{\text{nis}, \mathbb{A}^1} \Sigma^\infty \mathbb{P}^1)(k)$. We have to show that the image of $\tilde{\tau}$ in $\mathbf{K}_1^{\text{MW}}(k)$ under the map (2) is $[\zeta_p]$. For this, consider the following commutative diagram:

$$\begin{array}{ccccc}
\pi_2(L_p L_{\text{nis}, \mathbb{A}^1} \mathbb{P}^1)(k) & \xrightarrow{\text{blue}} & \pi_2(L_p K_{\text{nis}}(\mathbb{G}_m, 1))(k) & \xrightarrow{\cong} & \mathbb{Z}(1)(k) \\
\downarrow \text{red} & & \downarrow \cong & & \downarrow \text{blue} \\
\pi_2(L_p L_{\text{nis}, \mathbb{A}^1} \Sigma^\infty \mathbb{P}^1)(k) & \longrightarrow & \pi_2(L_p \Sigma H_{\text{nis}} \mathbb{G}_m)(k) & \xrightarrow{\cong} & \mathbb{Z}(1)(k) \\
\downarrow \text{red} & & \downarrow & & \downarrow \text{blue} \\
\pi_2(L_{\text{nis}, \mathbb{A}^1} \Sigma^\infty \mathbb{P}^1 // p)(k) & \longrightarrow & \pi_2(\Sigma H_{\text{nis}} \mathbb{G}_m // p)(k) & \xrightarrow{\cong} & \mu_p(k) \\
\downarrow \text{red} & & \downarrow & & \downarrow \text{blue} \\
\pi_1(L_{\text{nis}, \mathbb{A}^1} \Sigma^\infty \mathbb{P}^1)(k) & \longrightarrow & \pi_1(\Sigma H_{\text{nis}} \mathbb{G}_m)(k) & & \\
\downarrow \cong & & \downarrow \cong & & \\
\mathbf{K}_1^{\text{MW}}(k) & \xrightarrow{\text{red} \cong} & \mathbb{G}_m(k) & &
\end{array}$$

Here, the black horizontal maps on the left are induced by the map

$$L_{\text{nis}, \mathbb{A}^1} \Sigma^\infty \mathbb{P}^1 \rightarrow \Sigma^\infty K_{\text{nis}}(\mathbb{G}_m, 1) \rightarrow \tau_{\leq 1} \Sigma^\infty K_{\text{nis}}(\mathbb{G}_m, 1) \cong \Sigma H_{\text{nis}} \mathbb{G}_m,$$

where the last equivalence follows from the Freudenthal suspension theorem for ∞ -topoi, cf. [DH21, Corollary 4.16]. Note that the bottom **red** arrow is an equivalence as k is algebraically closed (see e.g. the short exact sequence in [Dég23, Corollary 2.3.10] and the fact that the fundamental ideal $I(k) = \ker(\text{GW}(k) \rightarrow \mathbb{Z})$ is zero if k is algebraically closed, as then $\text{GW}(k) \cong \mathbb{Z}$). By definition of τ the **blue** composition maps τ to ζ_p , which is exactly what we wanted to show for the **red** composition. \square

Now, in view of the construction of Bachmann's τ_{st} [Bac21a, Lemma 6.4] and the proof of [Bac21a, Theorem 6.5], we have the following result.

Theorem 7.6 (Bachmann). *Let $f: L_p \Sigma^2 \mathbb{S} \rightarrow L_p L_{\text{nis}, \mathbb{A}^1} \Sigma^\infty \mathbb{P}^1 \in \mathcal{SH}^{S^1}(k)$ be a map of stable τ -type. Then $L_p L_{\text{ét}, \mathbb{A}^1}(\sigma_{st} f)$ is an automorphism.*

Proposition 7.7. *The (p -completed) suspension spectrum functor*

$$L_p \Sigma^\infty: \text{Shv}_{\text{ét}}^h(\text{Sm}_S)_p^\wedge \rightarrow \text{Sp}(\text{Shv}_{\text{ét}}^h(\text{Sm}_S))_p^\wedge$$

induces an isomorphism of rings

$$\pi_0 \text{Map}(L_p \mathbb{S}^2, L_p \mathbb{S}^2) \rightarrow \pi_0 \text{Map}(L_p \Sigma^2 \mathbb{S}, L_p \Sigma^2 \mathbb{S}).$$

Proof. We first show that both sides are equivalent to \mathbb{Z}_p (as groups). Note that by adjunction, we have $\pi_0 \text{Map}(L_p S^2, L_p S^2) \cong \pi_0 \text{Map}(S^2, L_p S^2) \cong \pi_2(L_p S^2)(k)$. Now, from Lemma 2.37 we see that evaluation at k is a limit-preserving geometric morphism, and in particular commutes with homotopy objects and p -completion. Hence, the above is equivalent to $\pi_2 L_p(S^2(k)) = \pi_2(L_p S_{\text{top}}^2) \cong \mathbb{Z}_p$.

A similar calculation shows that the right-hand side is also \mathbb{Z}_p .

Since both abelian groups are endowed with ring structures (via composition), and since the group \mathbb{Z}_p has a unique ring structure (after a choice of unit $1 \in \mathbb{Z}_p$ which is given by the identity morphism), it follows that both groups are equivalent to \mathbb{Z}_p as rings. Since the map is induced by a functor, it is clear that it is a map of rings. But now the only ring map $\mathbb{Z}_p \rightarrow \mathbb{Z}_p$ is the identity. \square

From now on we will also write $\tau: L_p S^2 \rightarrow L_p L_{\text{ét}, \mathbb{A}^1} \mathbb{P}^1$ for $L_{\text{ét}, \mathbb{A}^1} \tau$.

Theorem 7.8. *The maps $\tau: L_p S^2 \rightarrow L_p L_{\text{ét}, \mathbb{A}^1} \mathbb{P}^1$ and $\sigma: L_p L_{\text{ét}, \mathbb{A}^1} \mathbb{P}^1 \rightarrow L_p S^2$ form a retract (up to precomposing τ with an automorphism of $L_p S^2$), i.e. there is a homotopy $\sigma\tau \cong \text{id}_{L_p S^2}$.*

Proof. By Proposition 7.7, the suspension spectrum functor induces an isomorphism of rings

$$\pi_0 \text{Map}(L_p S^2, L_p S^2) \rightarrow \pi_0 \text{Map}(L_p \Sigma^2 \mathbb{S}, L_p \Sigma^2 \mathbb{S}).$$

As by definition $L_p \Sigma^\infty \sigma \cong \sigma_{st}$, and $L_p \Sigma^\infty \tau$ is of stable τ -type by Proposition 7.5, we see that $L_p \Sigma^\infty(\sigma\tau)$ is an automorphism, cf. Theorem 7.6. Hence, in view of the ring isomorphism above, we see that also $\sigma\tau$ is an automorphism. Therefore, up to replacing τ by $\tau(\sigma\tau)^{-1}$, we see that τ and σ form a retract. \square

Actually, the retract constructed above is an equivalence after two suspensions. For this, we need the following simple lemma:

Lemma 7.9. *The swap endomorphism*

$$\text{swap}: L_{\text{nis}, \mathbb{A}^1} \mathbb{P}^1 \wedge L_{\text{nis}, \mathbb{A}^1} \mathbb{P}^1 \rightarrow L_{\text{nis}, \mathbb{A}^1} \mathbb{P}^1 \wedge L_{\text{nis}, \mathbb{A}^1} \mathbb{P}^1$$

is homotopic to the identity.

Proof. For readability, we will exclude the functor $L_{\text{nis}, \mathbb{A}^1}$ from the notation. The symmetric monoidal suspension spectrum functor $\Sigma_{\mathbb{P}^1}^\infty: \text{Spc}(k)_* \rightarrow \mathcal{SH}(k)$ induces a morphism on homotopy classes of maps

$$\Phi: [\mathbb{P}^1 \wedge \mathbb{P}^1, \mathbb{P}^1 \wedge \mathbb{P}^1] \rightarrow [\Sigma_{\mathbb{P}^1}^\infty \mathbb{P}^1 \otimes \Sigma_{\mathbb{P}^1}^\infty \mathbb{P}^1, \Sigma_{\mathbb{P}^1}^\infty \mathbb{P}^1 \otimes \Sigma_{\mathbb{P}^1}^\infty \mathbb{P}^1]$$

which maps the swap morphism to the swap morphism. It therefore suffices to show that Φ is injective, and the stable swap morphism is homotopic to the identity. Recall the computations of Morel of $[(\mathbb{P}^1)^{\wedge n}, (\mathbb{P}^1)^{\wedge n}] \cong \mathbf{K}_0^{\text{MW}}(k) = \text{GW}(k)$ for all $n \geq 2$, cf. [Mor12, Corollary 24]. Here, $\text{GW}(k)$ is the Grothendieck-Witt ring of k . This then stabilizes to show that Φ is in fact an isomorphism. In order to prove the second claim, we proceed as follows: under the equivalence

$$[\Sigma_{\mathbb{P}^1}^\infty \mathbb{P}^1 \otimes \Sigma_{\mathbb{P}^1}^\infty \mathbb{P}^1, \Sigma_{\mathbb{P}^1}^\infty \mathbb{P}^1 \otimes \Sigma_{\mathbb{P}^1}^\infty \mathbb{P}^1] \cong \text{GW}(k),$$

the swap morphism corresponds to the element $\langle -1 \rangle \in \text{GW}(k)$, for this combine the proof of [Mor03, Lemma 6.1.1 (2)] with [Mor03, Remark 6.3.5]. As k is algebraically closed, in particular $-1 \in k^\times$ is a square, and we see that $\langle -1 \rangle = 1 \in \text{GW}(k)$. \square

Theorem 7.10. *The map*

$$L_p L_{\text{ét}, \mathbb{A}^1} \Sigma^2 \tau: L_p L_{\text{ét}, \mathbb{A}^1} S^4 \cong L_p L_{\text{ét}, \mathbb{A}^1} \Sigma^2 S^2 \rightarrow L_p L_{\text{ét}, \mathbb{A}^1} \Sigma^2 \mathbb{P}^1$$

is an equivalence.

Proof. Consider the following diagram where we omit p -completions and étale- \mathbb{A}^1 -localizations:

$$\begin{array}{ccccc} S^2 \wedge \mathbb{P}^1 & \xrightarrow{\tau \wedge \mathbb{P}^1} & \mathbb{P}^1 \wedge \mathbb{P}^1 & \xrightarrow{\text{swap}} & \mathbb{P}^1 \wedge \mathbb{P}^1 \\ \downarrow S^2 \wedge \sigma & & \downarrow \mathbb{P}^1 \wedge \sigma & & \downarrow \sigma \wedge \mathbb{P}^1 \\ S^2 \wedge S^2 & \xrightarrow{\tau \wedge S^2} & \mathbb{P}^1 \wedge S^2 & \xrightarrow{\text{swap}} & S^2 \wedge \mathbb{P}^1 \\ \downarrow \text{swap} & & \downarrow \text{swap} & & \downarrow \text{swap} \\ S^2 \wedge S^2 & \xrightarrow{S^2 \wedge \tau} & S^2 \wedge \mathbb{P}^1 & & \end{array}$$

Here, the swap morphisms $\text{swap}: S^2 \wedge S^2 \rightarrow S^2 \wedge S^2$ and $\text{swap}: \mathbb{P}^1 \wedge \mathbb{P}^1 \rightarrow \mathbb{P}^1 \wedge \mathbb{P}^1$ are homotopic to the identity, for the first claim this holds since

$$\det \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = 1$$

and for the second claim see Lemma 7.9 (using that $L_{\text{ét}, \mathbb{A}^1}$ and L_p are symmetric monoidal). Note that the diagram commutes: the top left square by functoriality of \wedge , the bottom left and top right square since \wedge is symmetric monoidal, and the bottom right triangle is trivially commutative. Since $\sigma\tau = \text{id}$ by Theorem 7.8, it follows that the red composition on the top and right is the identity. Hence, also the blue composition on the left and bottom is the identity, proving that $S^2 \wedge \sigma$ is a two-sided inverse of $S^2 \wedge \tau$. \square

Remark 7.11. It is unknown to the author if already τ is an equivalence. On the other hand, the retract only exists on S^2 , i.e. there is no retract $L_p S^1 \rightarrow L_p \mathbb{G}_m \rightarrow L_p S^1$. Indeed, as S^1 is connected, so is $L_p S^1$, and in fact one can show that $L_p S^1 \cong K(\mathbb{Z}_p, 1)$. On the other hand, since \mathbb{G}_m is 0-truncated, we get that $L_p \mathbb{G}_m = \mathbb{G}_m$, cf. [Mat24b, Lemma 3.13]. In particular, $L_p \mathbb{G}_m$ is 0-truncated and hence $L_p S^1$ cannot be a retract of it.

8 Examples of étale \mathbb{A}^1 -nilpotent sheaves

In this section, we will give examples of p -completely étale \mathbb{A}^1 -nilpotent sheaves. We use those to prove our main theorem, Theorem A, in Corollary 8.8.

Lemma 8.1. *Let S be a scheme and $E \in \text{Sp}(\text{Shv}_{\text{ét}}^h(\text{Sm}_S))$. If $L_{\mathbb{Q}} E$ is \mathbb{A}^1 -invariant, and $E//p$ is \mathbb{A}^1 -invariant for all primes p , then also E is \mathbb{A}^1 -invariant.*

Proof. We first show that for every prime p , the p -completion $L_p E$ is \mathbb{A}^1 -invariant. Since $L_p E \cong \lim_n E//p^n$ and \mathbb{A}^1 -invariant sheaves are stable under limits, it suffices to show that $E//p^n$ is \mathbb{A}^1 -invariant for all n . We assumed this to be the case for $n = 1$. For $n > 1$, recall that there is a fiber sequence

$$E//p^n \rightarrow E//p^{n+1} \rightarrow E//p,$$

and we conclude by induction. Now consider the cartesian fracture square

$$\begin{array}{ccc} E & \longrightarrow & \prod_p L_p E \\ \downarrow & \lrcorner & \downarrow \\ L_{\mathbb{Q}} E & \longrightarrow & L_{\mathbb{Q}} \prod_p L_p E, \end{array}$$

see e.g. [Mat24a, Corollary 7.3]. By the above, all the $L_p E$ are \mathbb{A}^1 -invariant. Since \mathbb{A}^1 -invariant objects are stable under limits, this holds also for the product. Moreover, the rationalization of the \mathbb{A}^1 -invariant sheaf of spectra $F := \prod_p L_p E$ is again \mathbb{A}^1 -invariant: this holds since $L_{\mathbb{Q}} F$ is given by a filtered colimit of a diagram involving only F (cf. [Mat24a, Lemma 2.6]) and \mathbb{A}^1 -invariant sheaves of spectra are closed under colimits, cf. Lemma 3.5. On the other hand, $L_{\mathbb{Q}} E$ is \mathbb{A}^1 -invariant by assumption. Hence, so is E as a pullback of \mathbb{A}^1 -invariant sheaves of spectra. \square

Lemma 8.2. *Let k be an algebraically closed field, $X \in \mathrm{Sm}_k$ and $x \in X$ a point. Let $F \in \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(k_{\mathrm{\acute{e}t}})$ or $F \in \mathrm{Sp}(\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(k_{\mathrm{\acute{e}t}}))$. Then $(\iota_! F)(k) \cong (\iota_! F)(\mathcal{O}_{X,x}^{\mathrm{sh}})$.*

Proof. If k is algebraically closed, then $\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(k_{\mathrm{\acute{e}t}}) \cong \mathcal{A}n$ via evaluation at k by Lemma 2.36. In particular, we see that $\iota_!$ is equivalent to the constant sheaf functor (as $\mathcal{A}n$ is the initial ∞ -topos). In particular, $s^* \iota_! \cong \mathrm{id}_{\mathcal{A}n}$ for every point $s^*: \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(k_{\mathrm{\acute{e}t}}) \rightarrow \mathcal{A}n$. Since the points of the étale topos are given by strictly henselian local rings, cf. Proposition 2.18, the result follows. \square

Lemma 8.3. *Let k be a perfect field with $\mathrm{cd}(k) < \infty$ and write e for the exponential characteristic of k . Let $E \in \mathcal{SH}(k)$ be a motivic spectrum such that multiplication by e is invertible on E . Then $L_{\mathrm{\acute{e}t}} \omega^{\infty} E \in \mathrm{Sp}(\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_k))$ is \mathbb{A}^1 -invariant.*

Proof. Let \bar{k} be an algebraic closure of k . Write $\rho^*: \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_k) \rightarrow \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_{\bar{k}})$ for the geometric morphism from Definition 2.30. By Lemma 3.9 we see that ρ^* detects \mathbb{A}^1 -invariance, i.e. it suffices to show that $\rho^* L_{\mathrm{\acute{e}t}} \omega^{\infty} E$ is \mathbb{A}^1 -invariant. Consider the following diagram:

$$\begin{array}{ccccc} \mathrm{Sp}(\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_k)) & \xleftarrow{L_{\mathrm{\acute{e}t}}} & \mathrm{Sp}(\mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_k)) & \xleftarrow{\omega^{\infty}} & \mathcal{SH}(k) \\ \downarrow \rho^* & & \downarrow \rho^* & & \downarrow \rho^* \\ \mathrm{Sp}(\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_{\bar{k}})) & \xleftarrow{L_{\mathrm{\acute{e}t}}} & \mathrm{Sp}(\mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_{\bar{k}})) & \xleftarrow{\omega^{\infty}} & \mathcal{SH}(\bar{k}), \end{array}$$

where the functor in the middle is defined in the same way as the functor on the left, and the functor on the right is its \mathbb{G}_m -stabilization. The diagram commutes: for the left square this holds as all involved functors are induced by morphisms of sites, and they already commute on the level of sites. For the right square, it suffices to note that $\rho^*: \mathrm{Sp}(\mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_k)) \rightarrow \mathrm{Sp}(\mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_{\bar{k}}))$ commutes with \mathbb{G}_m -loops, i.e. $\Omega_{\mathbb{G}_m} \rho^* X \cong \rho^* \Omega_{\mathbb{G}_m} X$. For this, see the discussion before [Hoy13, Lemma A.7]. Therefore, $\rho^* L_{\mathrm{\acute{e}t}} \omega^{\infty} E \cong L_{\mathrm{\acute{e}t}} \omega^{\infty} \rho^* E$. Moreover, multiplication by e on $\rho^* E$ is clearly an equivalence. Hence, we may assume that $E \in \mathcal{SH}(\bar{k})$, i.e. that k is already algebraically closed.

Write $F := L_{\mathrm{\acute{e}t}} \omega^{\infty} E \in \mathrm{Sp}(\mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_k))$. By Lemma 8.1, it suffices to show that $L_{\mathbb{Q}} F$ is \mathbb{A}^1 -invariant and that the $F//p$ are \mathbb{A}^1 -invariant for all primes p .

We first show that $L_{\mathbb{Q}}F$ is \mathbb{A}^1 -invariant. Since $L_{\text{ét}}$ and ω^∞ commute with filtered colimits (see [BY20, Lemma 6.1] for the latter), it follows from [Mat24a, Corollary 2.7] that $L_{\mathbb{Q}}F \cong L_{\text{ét}}\omega^\infty L_{\mathbb{Q}}E$. We have seen in Proposition 4.1 that $\omega^\infty L_{\mathbb{Q}}E$ satisfies étale hyperdescent, hence we see that $L_{\text{ét}}\omega^\infty L_{\mathbb{Q}}E \cong \omega^\infty L_{\mathbb{Q}}E$, which is \mathbb{A}^1 -invariant.

We finish the proof by showing that $F//p$ is \mathbb{A}^1 -invariant. If $p = e$, then multiplication by p is invertible on F by assumption, and hence $F//p = 0$ is \mathbb{A}^1 -invariant. Hence, we may assume that $p \in k^\times$. Note that $F//p$ is p -complete, and hence by [BH21, Theorem 3.1] it suffices to show that $F//p$ is small, i.e. that $\iota_! \iota^*(F//p) \rightarrow F//p$ is an equivalence. Since we work in the étale topology, it hence suffices to show that $(\iota_! \iota^*(F//p))(\mathcal{O}_{X,x}^{sh}) \rightarrow (F//p)(\mathcal{O}_{X,x}^{sh})$ is an equivalence for every $X \in \text{Sm}_k$ and $x \in X$, cf. Proposition 2.18. In fact, by [Mil80, Chapter II, Remark 2.17 (b)] we may check this on closed points (we can use the 1-categorical result since we work with hypersheaves and hence can check equivalences on homotopy sheaves). So pick an $X \in \text{Sm}_k$ and $x \in X$ a closed point. This implies that $k(x)$ is a finite (hence algebraic) field extension of k . Since k is algebraically closed, it follows that $k = k(x)$. In particular, we see that $\mathcal{O}_{X,x}^{sh} \cong \mathcal{O}_{X,x}^h$. We first show that $(F//p)(\mathcal{O}_{X,x}^{sh}) \cong (F//p)(k)$. Indeed, since both $\mathcal{O}_{X,x}^{sh}$ and k are strictly henselian, and $F//p \cong L_{\text{ét}}\omega^\infty E//p$, it suffices to show that $(\omega^\infty E//p)(\mathcal{O}_{X,x}^{sh}) \cong (\omega^\infty E//p)(k)$. Since $\mathcal{O}_{X,x}^{sh} \cong \mathcal{O}_{X,x}^h$ this follows from [AD18, Theorem 1.2] as $p \in k^\times$ and multiplication by p^2 is null on $E//p$. Now consider the following equivalences

$$(F//p)(k) \cong (\iota_! \iota^*(F//p))(k) \cong (\iota_! \iota^*(F//p))(\mathcal{O}_{X,x}^{sh}).$$

Here we used that k is living in the small étale site and Lemma 8.2. Combining these equivalences yields $(F//p)(\mathcal{O}_{X,x}^{sh}) \cong (\iota_! \iota^*(F//p))(\mathcal{O}_{X,x}^{sh})$ which is what we wanted to show. \square

Proposition 8.4. *Let k be a perfect field with $\text{cd}(k) < \infty$ and write e for the exponential characteristic of k . Let $X \in \text{Spc}(k)_*$ be a nilpotent and 2-effective motivic space that is $\mathbb{Z}[\frac{1}{e}]$ -local. Then $L_{\text{ét}}X \in \text{Shv}_{\text{ét}}^h(\text{Sm}_k)_*$ is étale \mathbb{A}^1 -nilpotent. In particular, $L_{\text{ét}}X$ is still \mathbb{A}^1 -invariant.*

Proof. Choose motivic spaces X_n and \mathbb{P}^1 -infinite loop spaces $K_n = \Omega_{\mathbb{P}^1,*}^\infty E_n$ with $E_n \in \mathcal{SH}(k)$ a 2-connective 2-effective motivic spectrum as in Proposition 2.14. We may moreover assume that multiplication by e is invertible on E_n by Lemma 2.16.

As $L_{\text{ét}}$ is a geometric morphism, it preserves finite limits, and hence we get fiber sequences $L_{\text{ét}}X_{n+1} \rightarrow L_{\text{ét}}X_n \rightarrow L_{\text{ét}}K_n$, where $L_{\text{ét}}K_n \cong L_{\text{ét}}\Omega_{\mathbb{P}^1,*}^\infty E_n \cong \Omega_*^\infty L_{\text{ét}}\omega^\infty E_n$ is a connected infinite loop sheaf. We moreover have $L_{\text{ét}}X \cong \lim_n L_{\text{ét}}X_n$: for this, note that as the connectivity of the K_i tends to ∞ as $i \rightarrow \infty$, the tower $(X_n)_n$ is highly connected. Hence, the claim follows from Proposition 2.3, using that the étale hypersheafification functor $L_{\text{ét}}$ upgrades to a morphism of ∞ -topoi with locally finite dimensional cover, cf. Proposition 2.26. Hence, we are reduced to show that $L_{\text{ét}}\omega^\infty E_n$ is \mathbb{A}^1 -invariant. This was shown in Lemma 8.3.

It follows now from Lemma 3.13 that $L_{\text{ét}}X$ is in particular \mathbb{A}^1 -invariant. \square

Lemma 8.5 (Stability under retracts). *Let $X, Y \in \text{Shv}_{\text{ét}}^h(\text{Sm}_k)_*$. Suppose that there exists a retract $L_p X \xrightarrow{i} L_p Y \xrightarrow{r}$. If Y is p -completely small, then so is X .*

Proof. By assumption $L_p Y \cong L_p \iota_! \iota^* Y$, and we have to show that $L_p X \cong L_p \iota_! \iota^* X$. For this, consider the commutative diagram

$$\begin{array}{ccccc} L_p \iota_! \iota^* L_p X & \xleftarrow{\cong} & L_p \iota_! \iota^* X & \longrightarrow & L_p X \\ \downarrow i & & & & \downarrow i \\ L_p \iota_! \iota^* L_p Y & \xleftarrow{\cong} & L_p \iota_! \iota^* Y & \xrightarrow{\cong} & L_p Y. \end{array}$$

Since $\iota_!$ and ι^* preserve p -equivalences (as they are left adjoints), the left horizontal maps are equivalences. By assumption the vertical maps admit retractions, and the lower right horizontal map is an equivalence. As equivalences are stable under retracts, we conclude that the upper right horizontal map is an equivalence. This is precisely what we wanted to prove. \square

We now use the last few lemmas and the retract from Theorem 7.8 to show that in fact over a perfect field k any \mathbb{A}^1 -invariant 4-connective étale sheaf is p -completely small. We start with the case of an algebraically closed field and then deduce the general case.

Corollary 8.6. *Let k be an algebraically closed field, $p \neq \text{char}(k)$ a prime and $X \in \text{Shv}_{\text{ét}}^h(\text{Sm}_k)_*$. If X is 4-connective and \mathbb{A}^1 -invariant then X is p -completely small, i.e. $L_p X \cong L_p \iota_! \iota^* X$.*

Proof. By Proposition 2.38 there is an equivalence $X \cong \text{colim}_i W_i$, with W_i of the form $W_i \cong \Sigma^4 V_i$ for some $V_i \in \text{Shv}_{\text{ét}}^h(\text{Sm}_k)_*$.

Suppose for the moment that we know that $L_{\text{ét}, \mathbb{A}^1} W_i$ is p -completely small. We show that the same is true for X . For this, note that we have maps

$$X = \text{colim}_i W_i \rightarrow \text{colim}_i L_{\text{ét}, \mathbb{A}^1} W_i \rightarrow L_{\text{ét}, \mathbb{A}^1} \text{colim}_i L_{\text{ét}, \mathbb{A}^1} W_i.$$

Since \mathbb{A}^1 -equivalences are stable under colimits, and since X is \mathbb{A}^1 -invariant, we see that

$$L_{\text{ét}, \mathbb{A}^1} \text{colim}_i L_{\text{ét}, \mathbb{A}^1} W_i \cong L_{\text{ét}, \mathbb{A}^1} \text{colim}_i W_i \cong L_{\text{ét}, \mathbb{A}^1} X \cong X.$$

In particular, by construction we therefore have a retract diagram

$$X \rightarrow \text{colim}_i L_{\text{ét}, \mathbb{A}^1} W_i \rightarrow X.$$

By assumption the map $\iota_! \iota^* L_{\text{ét}, \mathbb{A}^1} W_i \rightarrow L_{\text{ét}, \mathbb{A}^1} W_i$ is a p -equivalence for all i , and since both $\iota_!$ and ι^* commute with colimits, and p -equivalences are stable under colimits, we see that the canonical map $\iota_! \iota^* \text{colim}_i L_{\text{ét}, \mathbb{A}^1} W_i \rightarrow \text{colim}_i L_{\text{ét}, \mathbb{A}^1} W_i$ is a p -equivalence. Hence, also the retract $\iota_! \iota^* X \rightarrow X$ is a p -equivalence, i.e. X is p -completely small.

It remains to show that $L_{\text{ét}, \mathbb{A}^1}(S^4 \wedge Z)$ is p -completely small for every $Z \in \text{Shv}_{\text{ét}}^h(\text{Sm}_k)_*$. By applying Theorem 7.8 twice (here we use our assumption that k is algebraically closed), there is a retract

$$L_p L_{\text{ét}, \mathbb{A}^1}(S^4 \wedge Z) \rightarrow L_p L_{\text{ét}, \mathbb{A}^1}(\mathbb{P}^1 \wedge \mathbb{P}^1 \wedge Z) \cong L_p L_{\text{ét}, \mathbb{A}^1}(S^2 \wedge \mathbb{G}_m^{\wedge 2} \wedge Z).$$

Write $Y := L_{\mathbb{Z}[\frac{1}{e}]} L_{\text{nis}, \mathbb{A}^1}(S^2 \wedge \mathbb{G}_m^{\wedge 2} \wedge \iota_{\text{nis}} Z) \in \text{Spc}(k)_*$ for the Nisnevich local version (where we also invert the exponential characteristic e of k). This is clearly

2-connective and 2-effective. Hence, we see that $L_{\text{ét}}Y$ is étale \mathbb{A}^1 -nilpotent, cf. Proposition 8.4, and in particular \mathbb{A}^1 -invariant by Lemma 3.13, whence $L_{\mathbb{Z}[\frac{1}{e}]}L_{\text{ét}, \mathbb{A}^1}(S^2 \wedge \mathbb{G}_m^{\wedge 2} \wedge Z) \cong L_{\text{ét}}Y$. Thus, $L_p L_{\text{ét}, \mathbb{A}^1}(S^4 \wedge Z)$ is a retract of $L_p L_{\text{ét}}Y$ (for this note that $L_p L_{\mathbb{Z}[\frac{1}{e}]} \cong L_p$ as functors since $p \neq e$ by assumption, and p -completion in particular inverts $\mathbb{Z}[\frac{1}{e}]$ -local equivalences). Since by Lemma 3.11 $L_{\text{ét}}Y$ is also p -completely étale \mathbb{A}^1 -nilpotent, it is in particular p -completely small by Theorem 6.6. Hence, we see that the same is true for $L_{\text{ét}, \mathbb{A}^1}(S^4 \wedge Z)$ using Lemma 8.5. \square

We now generalize the above corollary to arbitrary perfect fields of finite étale cohomological dimension.

Proposition 8.7. *Let k be a perfect field with $\text{cd}(k) < \infty$, $p \neq \text{char}(k)$ a prime and $X \in \text{Shv}_{\text{ét}}^h(\text{Sm}_k)_*$. If X is 4-connective and \mathbb{A}^1 -invariant, then X is p -completely small, i.e. $L_p \iota_! \iota^* X \cong L_p X$.*

Proof. Let \bar{k} be an algebraic closure of k . Consider the geometric morphisms $\rho^*: \text{Shv}_{\text{ét}}^h(\text{Sm}_k) \rightleftarrows \text{Shv}_{\text{ét}}^h(\text{Sm}_{\bar{k}}) : \rho_*$ and $\rho^*: \text{Shv}_{\text{ét}}^h(k_{\text{ét}}) \rightleftarrows \text{Shv}_{\text{ét}}^h(\bar{k}_{\text{ét}}) : \rho_*$ from Definition 2.30, both left adjoints are conservative by Lemma 2.35. Moreover, $\iota^* \rho^* \cong \rho^* \iota^*$, as well as $\iota_! \rho^* \cong \rho^* \iota_!$ by Lemma 2.34. As ρ^* is conservative, it suffices to show by [Mat24b, Lemma 3.11] that $\rho^* \iota_! \iota^* X \rightarrow \rho^* X$ is a p -equivalence. By the above, this is equivalent to the morphism

$$\iota_! \iota^* \rho^* X \rightarrow \rho^* X.$$

As $\rho^* X$ is 4-connective (since ρ^* is a geometric morphism) and \mathbb{A}^1 -invariant by Lemma 3.8, it follows from Corollary 8.6 that $\iota_! \iota^* \rho^* X \rightarrow \rho^* X$ is a p -equivalence. \square

Corollary 8.8. *Let k be a perfect field with $\text{cd}(k) < \infty$ and $p \neq \text{char}(k)$. Then*

$$\iota_!^p : (\text{Shv}_{\text{ét}}^h(k_{\text{ét}})_*)_p^\wedge \rightarrow (\text{Shv}_{\text{ét}}^h(\text{Sm}_k)_*)_p^\wedge$$

induces an equivalence between the full subcategory of $(\text{Shv}_{\text{ét}}^h(k_{\text{ét}})_)_p^\wedge$ consisting of those sheaves that are the p -completion of a 4-connective sheaf, and the full subcategory of $(\text{Shv}_{\text{ét}}^h(\text{Sm}_k)_*)_p^\wedge$ consisting of those sheaves that are the p -completion of a 4-connective \mathbb{A}^1 -invariant sheaf.*

Proof. From Lemma 6.3 we see that $\iota_!^p$ is fully faithful. First, let $Y \in \text{Shv}_{\text{ét}}^h(k_{\text{ét}})_{*, \geq 4}$ be 4-connective. Note that $\iota_!^p L_p Y \cong L_p \iota_! Y$ (since $\iota_!$ preserves p -equivalences as a left adjoint), and by Proposition 6.4 this sheaf is \mathbb{A}^1 -invariant. Moreover, $\iota_! Y$ is 4-connective since $\iota_!$ is a geometric morphism.

To prove the corollary, it thus suffices to show that if $X \in \text{Shv}_{\text{ét}}^h(\text{Sm}_k)_{*, \geq 4}$ is 4-connective and \mathbb{A}^1 -invariant, then $L_p X$ is in the essential image of $\iota_!^p$ restricted to the full subcategory of $(\text{Shv}_{\text{ét}}^h(k_{\text{ét}})_*)_p^\wedge$ consisting of those sheaves that are the p -completion of a 4-connective sheaf. By Proposition 8.7 we see that $L_p X \cong L_p \iota_! \iota^* X \cong \iota_!^p L_p \iota^* X$. Since ι^* is a geometric morphism, we see that $\iota^* X$ is 4-connective. \square

9 Application: Étale strict \mathbb{A}^1 -invariance

In this section, we use our rigidity result to prove a weak version of Morel's theorem that strongly \mathbb{A}^1 -invariant Nisnevich sheaves of abelian groups are strictly \mathbb{A}^1 -invariant. For this, we need the following list of definitions.

Definition 9.1. Let S be a scheme. Let $A \in \mathrm{Shv}_{\mathrm{\acute{e}t}}(\mathrm{Sm}_S, \mathcal{A}b)$ be an étale sheaf of abelian groups. We say that A is

- *n -strictly étale \mathbb{A}^1 -invariant* for some $n \in \mathbb{N}$ if $K(A, n) \in \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)$ is \mathbb{A}^1 -invariant,
- *strictly étale \mathbb{A}^1 -invariant* if A is n -strictly étale \mathbb{A}^1 -invariant for all $n \in \mathbb{N}$,
- *p -completely n -strictly étale \mathbb{A}^1 -invariant* for a prime p and $n \in \mathbb{N}$ if $L_p K(A, n) \in \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)$ is \mathbb{A}^1 -invariant,
- *p -completely strictly étale \mathbb{A}^1 -invariant* for a prime p if A is p -completely n -strictly étale \mathbb{A}^1 -invariant for all $n \in \mathbb{N}$,
- *rationally n -strictly étale \mathbb{A}^1 -invariant* for some $n \in \mathbb{N}$ if $L_{\mathbb{Q}} K(A, n) \in \mathrm{Shv}_{\mathrm{\acute{e}t}}^h(\mathrm{Sm}_S)$ is \mathbb{A}^1 -invariant,
- *rationally strictly étale \mathbb{A}^1 -invariant* if A is rationally n -strictly étale \mathbb{A}^1 -invariant for all $n \in \mathbb{N}$,

Lemma 9.2. *Let S be a scheme. Let $A \in \mathrm{Shv}_{\mathrm{\acute{e}t}}(\mathrm{Sm}_S, \mathcal{A}b)$ be an étale sheaf of abelian groups, and $1 \leq n \leq m$. If A is m -strictly étale \mathbb{A}^1 -invariant (resp. p -completely, resp. rationally), then A is n -strictly étale \mathbb{A}^1 -invariant (resp. p -completely, resp. rationally).*

Proof. We have $K(A, n) \cong \Omega^{m-n} K(A, m)$. As limits of \mathbb{A}^1 -invariant sheaves are \mathbb{A}^1 -invariant, the statement follows. For the p -complete or rational versions of the statement, note that $L_p K(A, n) \cong \tau_{\geq 1} \Omega^{m-n} L_p K(A, m)$, and $L_{\mathbb{Q}} K(A, n) \cong \tau_{\geq 1} \Omega^{m-n} L_{\mathbb{Q}} K(A, m)$ by Proposition 2.8 and [Mat24a, Lemma 3.13]. To show that these sheaves are \mathbb{A}^1 -invariant, it suffices to note that the connected cover functor preserves \mathbb{A}^1 -invariance, cf. Proposition 3.7. \square

Lemma 9.3. *Let S be a qcqs scheme. Let $A \in \mathrm{Shv}_{\mathrm{\acute{e}t}}(\mathrm{Sm}_S, \mathcal{A}b)$ be an étale sheaf of abelian groups, and $n \geq 1$. If A is n -strictly étale \mathbb{A}^1 -invariant, then A is rationally n -strictly étale \mathbb{A}^1 -invariant.*

If $n \geq 2$, and A is n -strictly étale \mathbb{A}^1 -invariant, then A is p -completely $(n-1)$ -strictly étale \mathbb{A}^1 -invariant for all primes p .

Proof. By assumption, $K(A, n)$ is \mathbb{A}^1 -invariant, we have to show that the same is true for $L_p K(A, n-1)$ and $L_{\mathbb{Q}} K(A, n)$.

We first prove that $L_{\mathbb{Q}} K(A, n)$ is \mathbb{A}^1 -invariant. Note that we have equivalences

$$\begin{aligned} L_{\mathbb{Q}} K(A, n) &\cong \Omega_*^{\infty} L_{\mathbb{Q}} \Sigma^n H A \\ &\cong \Omega_*^{\infty} \mathrm{colim}_{\mathbb{N}} \Sigma^n H A \\ &\cong \mathrm{colim}_{\mathbb{N}} \Omega_*^{\infty} \Sigma^n H A \\ &\cong \mathrm{colim}_{\mathbb{N}} K(A, n), \end{aligned}$$

where the colimit is over the \mathbb{N} -indexed diagram from [Mat24a, Lemma 2.6]. In the third equivalence, we used that Ω_*^∞ commutes with filtered colimits (combine [Lur09, Corollary 5.3.6.10] with the fact that filtered colimits commute with finite limits in any ∞ -topos, cf. [Lur09, Example 7.3.4.7]). Since \mathbb{A}^1 -invariant sheaves are stable under filtered colimits by Lemma 3.5, and $K(A, n)$ is \mathbb{A}^1 -invariant by assumption, the result follows.

We now show that if $n \geq 2$ also $L_p K(A, n-1)$ is \mathbb{A}^1 -invariant. For this, note that we have equivalences

$$\begin{aligned} L_p K(A, n-1) &\cong \tau_{\geq 1} \Omega_*^\infty L_p \Sigma^{n-1} H A \\ &\cong \tau_{\geq 1} \Omega_*^\infty \lim_k \Sigma^{n-1} H A // p^k \\ &\cong \tau_{\geq 1} \lim_k \Omega_*^\infty \Sigma^{n-1} H A // p^k \\ &\cong \tau_{\geq 1} \lim_k \text{fib} \left(K(A, n) \xrightarrow{p^k} K(A, n) \right). \end{aligned}$$

Here we used [Mat24b, Lemmas 3.17 and 2.5] in the first and second equivalence, that Ω_*^∞ preserves limits as a right adjoint in the third equivalence, and that

$$\Sigma^{n-1} H A // p^k \cong \text{fib} \left(\Sigma^n H A \xrightarrow{p^k} \Sigma^n H A \right)$$

in the last equivalence. Hence, as \mathbb{A}^1 -invariant sheaves are stable under limits and connected covers (cf. Proposition 3.7 for the latter), it suffices to show that $K(A, n)$ is \mathbb{A}^1 -invariant, which holds by assumption. \square

Proposition 9.4. *Let k be a perfect field with $\text{cd}(k) < \infty$ and $p \neq \text{char}(k)$ be a prime. Let $A \in \text{Shv}_{\text{ét}}(\text{Sm}_k, \mathcal{A}b)$ be an étale sheaf of abelian groups. Suppose that A is m -strictly étale \mathbb{A}^1 -invariant for some $m \geq 4$. Then A is p -completely strictly étale \mathbb{A}^1 -invariant, and $L_p H A$ is an \mathbb{A}^1 -invariant sheaf of spectra.*

Proof. It suffices to show that $L_p H A$ is \mathbb{A}^1 -invariant. Indeed, then for any $n \geq 1$ we have $L_p K(A, n) \cong \tau_{\geq 1} \Omega_*^\infty \Sigma^n L_p H A$ by [Mat24b, Corollary 3.18]. Hence, it is \mathbb{A}^1 -invariant as both Ω_*^∞ and $\tau_{\geq 1}$ preserve \mathbb{A}^1 -invariant sheaves, cf. Lemma 3.2 and Proposition 3.7.

Now, it is enough to show that $\iota_! \iota^* H A \rightarrow H A$ is a p -equivalence, as then $L_p H A \cong L_p \iota_! \iota^* H A$ is \mathbb{A}^1 -invariant by the proof of [Bac21a, Corollary 6.2]. For this, it is clearly enough to show that $\Sigma^m \iota_! \iota^* H A \rightarrow \Sigma^m H A$ is a p -equivalence. Since Ω_*^∞ commutes with both $\iota_!$ and ι^* , and detects p -equivalences between 1-connective objects by Lemma 2.5, it thus suffices to show that $\iota_! \iota^* \Omega_*^\infty \Sigma^m H A \rightarrow \Omega_*^\infty \Sigma^m H A$ is a p -equivalence. Since $\Omega_*^\infty \Sigma^m H A \cong K(A, m)$ and $m \geq 4$, this follows from Proposition 8.7. \square

Proposition 9.5. *Let k be a perfect field. Let $A \in \text{Shv}_{\text{ét}}(\text{Sm}_k, \mathcal{A}b)$ be an étale sheaf of abelian groups. Suppose that A is rationally 1-strictly étale \mathbb{A}^1 -invariant. Then A is rationally strictly étale \mathbb{A}^1 -invariant.*

Proof. First note that for every $n \geq 1$ we have $L_{\mathbb{Q}} K(A, n) \cong K(A_{\mathbb{Q}}, n)$ using e.g. [Mat24a, Proposition 3.12]. It follows from [V⁺00, Proposition 5.27] that $K_{\text{nis}}(A_{\mathbb{Q}}, n) \cong K_{\text{ét}}(A_{\mathbb{Q}}, n)$. But if $K_{\text{nis}}(A_{\mathbb{Q}}, 1)$ is \mathbb{A}^1 -invariant, then so is $K_{\text{nis}}(A_{\mathbb{Q}}, n)$ by Morel's theorem [Bac24, Theorem 1.7]. \square

Proposition 9.6. *Let k be a perfect field with $p = \text{char}(k) > 0$, and $A \in \text{Shv}_{\text{ét}}(\text{Sm}_k, \mathcal{A}b)$ be an étale sheaf of abelian groups. Suppose that A is p -completely 3-strictly étale \mathbb{A}^1 -invariant. Then $L_p H A \cong 0$ and $L_p K(A, n) = *$ for all $n \geq 1$. In particular, A is p -completely strictly étale \mathbb{A}^1 -invariant, and $L_p H A$ is \mathbb{A}^1 -invariant.*

Proof. This is a reformulation of Corollary 5.4. \square

Theorem 9.7. *Let k be a perfect field with $\text{cd}(k) < \infty$. Let $A \in \text{Shv}_{\text{ét}}(\text{Sm}_k, \mathcal{A}b)$ be an étale sheaf of abelian groups. Assume that A is 4-strictly étale \mathbb{A}^1 -invariant. Then A is strictly étale \mathbb{A}^1 -invariant.*

Proof. We have to show that A is m -strictly étale \mathbb{A}^1 -invariant for every $m \geq 1$. From [Mat24a, Theorem 8.7] we have a pullback square

$$\begin{array}{ccc} K(A, m) & \longrightarrow & \tau_{\geq 1} \prod_p L_p K(A, m) \\ \downarrow & \lrcorner & \downarrow \\ L_{\mathbb{Q}} K(A, m) & \longrightarrow & L_{\mathbb{Q}} \tau_{\geq 1} \prod_p L_p K(A, m). \end{array}$$

We have to show that the top left object is \mathbb{A}^1 -invariant. As limits of \mathbb{A}^1 -invariant sheaves are \mathbb{A}^1 -invariant, it suffices to show that the other objects in the above diagram are \mathbb{A}^1 -invariant. That $L_p K(A, m)$ is \mathbb{A}^1 -invariant for every p was shown in Proposition 9.4 if $p \neq \text{char}(k)$, and in Proposition 9.6 if $p = \text{char}(k)$, using that $L_p K(A, 3)$ is \mathbb{A}^1 -invariant by Lemma 9.3. Moreover, that $L_{\mathbb{Q}} K(A, m)$ is \mathbb{A}^1 -invariant is exactly Proposition 9.5, again using Lemma 9.3. Now also $\tau_{\geq 1} \prod_p L_p K(A, m)$ is \mathbb{A}^1 -invariant as a connected cover of a limit of \mathbb{A}^1 -invariant sheaves, cf. Proposition 3.7. For the object in the bottom right corner we compute

$$\begin{aligned} L_{\mathbb{Q}} \tau_{\geq 1} \prod_p L_p K(A, m) &\cong L_{\mathbb{Q}} \tau_{\geq 1} \prod_p \tau_{\geq 1} \Omega_*^{\infty} L_p \Sigma^m H A && [\text{Mat24b, Corollary 3.18}] \\ &\cong L_{\mathbb{Q}} \tau_{\geq 1} \prod_p \Omega_*^{\infty} L_p \Sigma^m H A && [\text{Mat24a, Lemma 4.2}] \\ &\cong L_{\mathbb{Q}} \tau_{\geq 1} \Omega_*^{\infty} \prod_p L_p \Sigma^m H A && (\Omega_*^{\infty} \text{ is a right adjoint}) \\ &\cong \tau_{\geq 1} \Omega_*^{\infty} L_{\mathbb{Q}} \prod_p L_p \Sigma^m H A && [\text{Mat24a, Lemma 3.15}]. \end{aligned}$$

Now $L_p H A$ is \mathbb{A}^1 -invariant (again by Proposition 9.4 if $p \neq \text{char}(k)$, and by Proposition 9.6 if $p = \text{char}(k)$), and hence so is the product over all primes. Since Ω_*^{∞} and $\tau_{\geq 1}$ preserve \mathbb{A}^1 -invariant sheaves by Lemma 3.2 and Proposition 3.7, it thus suffices to show that $L_{\mathbb{Q}}$ on $\text{Sp}(\text{Shv}_{\text{ét}}(\text{Sm}_k))$ preserves \mathbb{A}^1 -invariant sheaves of spectra. This holds since $L_{\mathbb{Q}} E$ is given by a filtered colimit of a diagram involving only E (cf. [Mat24a, Lemma 2.6]), and \mathbb{A}^1 -invariant sheaves of spectra are closed under colimits, cf. Lemma 3.5. This proves the theorem. \square

A Nilpotent morphisms

Let \mathcal{X} be an ∞ -topos. In this section we discuss nilpotent morphisms in \mathcal{X} . We essentially copy the contents from [AFH22] about nilpotent morphisms of motivic spaces to the setting of an ∞ -topos.

Definition A.1. Let $f: E \rightarrow B$ be a morphism in \mathcal{X}_* . We say that f is *nilpotent* if the fiber $\text{fib}(f)$ is connected, and the action of $\pi_1(E)$ on $\pi_n(\text{fib}(f))$ is nilpotent for every n .

Remark A.2. Let $X \in \mathcal{X}_*$. Then X is nilpotent if and only if $X \rightarrow *$ is nilpotent.

Lemma A.3. Let $q: E_2 \rightarrow E_1$ and $p: E_1 \rightarrow E_0$ be two morphisms in \mathcal{X}_* , such that $\text{fib}(q)$, $\text{fib}(p)$ and $\text{fib}(pq)$ are all connected. If p and q are nilpotent, then so is pq .

Proof. The proof of [AFH22, Proposition 3.3.2] can also be used for the ∞ -topos case. \square

Corollary A.4. Let $f: E \rightarrow B$ be a morphism in \mathcal{X}_* . Suppose that E and B are nilpotent, and that $\text{fib}(f)$ is connected. Then also f is nilpotent.

Proof. This is an application of Lemma A.3 for $q = f: E \rightarrow B$ and $p: B \rightarrow *$, using Remark A.2 and the assumption that B , E and $\text{fib}(f)$ are connected. \square

Proposition A.5 (Principal Moore–Postnikov tower). Let $f: E \rightarrow B$ be a morphism in \mathcal{X}_* . If f is nilpotent, then there is a system of sheaves E_i under E and over B with $E_0 \cong B$, such that $E \cong \lim_i E_i$, and such that there are fiber sequences

$$E_{i+1} \rightarrow E_i \rightarrow K(A_i, n_i)$$

with $A_i \in \text{Ab}(\text{Disc}(\mathcal{X}))$ and $n_i \geq 2$, such that $n_i \rightarrow \infty$ as $i \rightarrow \infty$.

Proof. The proof of [AFH22, Corollary 4.2.4] can also be used for the ∞ -topos case. \square

Lemma A.6. Let $f: E \rightarrow B$ be a morphism in \mathcal{X}_* . If f is nilpotent, and $\text{fib}(f)$ is k -connective for some $k \geq 2$, then we may assume that in the situation of Proposition A.5 the integers n_i are $\geq k$.

Proof. This is immediate from the construction. \square

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