MONADIC RESOLUTIONS FOR GENERALIZED SPACES

TOM BACHMANN, ANTON ENGELMANN, AND KLAUS MATTIS

ABSTRACT. We extend the work of Bousfield and Kan [BK72] on monadic resolutions of spaces to ∞-topoi, with applications to genuine G-equivariant spaces (G a finite group) and motivic spaces over a perfect field. In particular, we give a proof of the principal fibration lemma in this context. We apply the principal fibration lemma to prove convergence of several kinds of monadic resolutions in unstable equivariant and motivic homotopy theory. For example, we show that, over an algebraically closed field, the unstable Adams–Novikov spectral sequence (i.e., the monadic resolution corresponding to the algebraic cobordism spectrum MGL) converges for all nilpotent, connected, 2-effective motivic spaces.

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1. Introduction

Unstable homotopy theory is concerned with the study of spaces and their maps. Because spaces are nonlinear and wildly complicated, a standard strategy is to pass to linear invariants that are easier to compute and compare, such as homology. An evident question arises, which is *how much information is lost in this process?*, or *how much of a space can we recover from its homology?* When taking into account enough structure on the homology, there is a natural "best possible approximation" we can extract from the homology, known as the *Bousfield–Kan completion* (of a space at a homology theory). In their remarkable work [BK72], Bousfield and Kan identified essentially two situations in which the Bousfield–Kan completion can be described somewhat explicitly:

Theorem (Bousfield–Kan). *Let $X \in \mathcal{S}\mathrm{pc}_*$ be a pointed nilpotent space.*
- *If p is a prime and the homology theory is $H\mathbb{F}_p$, then the corresponding Bousfield–Kan completion coincides with the unstable p-completion.*
- *If $S \subset \mathbb{Z}$ is a multiplicative subset and the homology theory is $H\mathbb{Z}[S^{-1}]$, then the Bousfield–Kan completion coincides with the unstable S-localization.*

For example, the Bousfield–Kan completion of any nilpotent space at $H\mathbb{Z}$ recovers the space itself.

The aim of this article is to provide a treatment of the ideas of Bousfield–Kan in modern language, and apply them to other kinds of spaces. In particular, we will prove a Bousfield–Kan completion theorem for categories of sheaves as well as genuine equivariant and motivic spaces.

1.1. **Digression: categorical formulation.** Let

$$L\colon \mathcal{C} \rightleftarrows \mathcal{D} \colon R$$

be an adjunction between presentable ∞-categories. Suppose we attempt to study the ∞-category $\mathcal{C}$ through the lens of the functor L. A first observation is that if $X \in \mathcal{C}$, then the object $LX \in \mathcal{D}$ carries additional structure, e.g., the (adjunction) map $L(RL)X \simeq (LR)LX \to LX$. In a more structured way, we can observe that $C := LR\colon \mathcal{D} \to \mathcal{D}$ is a comonad, and LX is a coalgebra for the comonad C (see [Lur17, the dual of Proposition 4.7.3.3] and the discussion thereafter). A natural question would be to ask if X can be recovered from LX with this additional structure. It is clear that this is not possible in general, as we now explain. An L-equivalence in $\mathcal{C}$ is a map inverted by L. These form a strongly saturated class of small generation, and so, there is an associated Bousfield localization functor $\hat{L}\colon \mathcal{C} \to \mathcal{C}$ [Lur09, §5.5.4]. We can thus factor our starting adjunction as

$$\mathcal{C} \rightleftarrows \hat{L}\mathcal{C} \rightleftarrows \mathrm{Coalg}_C(\mathcal{D}) \rightleftarrows \mathcal{D},$$

where the first adjunction is a Bousfield localization and the third is a forgetful/cofree adjunction. Clearly the best we can hope for is that the middle adjunction is fully faithful, perhaps on some suitable subcategory of $\hat{L}\mathcal{C}$.

Remark 1.1. Note that the composite

$$\mathcal{C} \xrightarrow{\hat{L}} \mathrm{Coalg}_C(\mathcal{D}) \to \mathcal{C}$$

(where the second functor is right adjoint to the first) is equivalent to the *cobar construction* (also called the *Bousfield–Kan completion*)

$$X \mapsto \mathrm{CB}(X) := \lim_{[n]\in\Delta} (RL)^{n+1}X.$$

By construction, this object is L-local and receives a map from X. So we can rephrase our initial question as follows: for which $X \in \mathcal{C}$ is the canonical map

$$\hat{L}X \to \mathrm{CB}(X)$$

an equivalence?

1.2. **Results.** Our results fit into the categorical framework of the previous section. Typically, in addition to $\mathcal{C} \rightleftarrows \mathcal{D}$ we pick a ring $A \in \mathrm{CAlg}(\mathcal{D})$ and consider the composite adjunction

$$\mathcal{C} \rightleftarrows \mathcal{M}\mathrm{od}_A(\mathcal{D}).$$

We denote the resulting Bousfield localization by $\hat{L}_A$. Our goal is to exhibit (ideally many) examples of $X \in \mathcal{C}$ such that $\hat{L}_A$ coincides with the Bousfield–Kan completion of X (for the above adjunction). Moreover, we seek to identify $\hat{L}_A$ in more familiar terms. In order to do this, we prove results about Bousfield–Kan completions in a relatively general context. For a summary, see Section 1.4. We then apply these general results in settings of interest, and combine them with additional assumptions and techniques

specific to the settings, to obtain useful criteria in these specific situations. Here we concentrate on explaining the latter.

1.2.1. *Sheaves.* Our first application is the case where $\mathcal{C} = \mathcal{X}$ is an appropriate ∞-topos and $\mathcal{D} = \mathrm{Sp}(\mathcal{X})$ is its stabilization. Recall that in this case there are notions of *unstable n-completion* and *unstable S-localization*, see Section C, [Mat24b, Section 3] and [Mat24a, Section 3]. Generalizing the situation for classical spaces, these are given by Bousfield localization endofunctors

$$L_S, L_n \colon \mathcal{X} \to \mathcal{X}.$$

We can only understand these functors to a reasonable degree on *nilpotent sheaves*, i.e., sheaves built out of infinite loop sheaves in a controlled manner [Mat24b, Appendix A.2].

Theorem A (Sheaves, Theorem 6.8)**.** *Let $\mathcal{X}$ be an ∞-topos admitting a locally finite-dimensional cover* [Mat24a, Definition 5.1] *(see Example 6.4 for examples), let $A \in \mathrm{CAlg}(\mathrm{Sp}(\mathcal{X})_{\geq 0})$, and let $X \in \mathcal{X}_*$ be nilpotent. Then, depending on $\underline{\pi}_0(A)$ we can identify $\hat{L}_A X$ as:*

$\underline{\pi}_0(A)$	Localization functor $\hat{L}_A$
$S^{-1}\underline{\mathbb{Z}} \subset \underline{\mathbb{Q}}$	$L_S X$
$\underline{\mathbb{Z}}/n$	$L_n X$

In these cases, we can moreover compute the localization of X with respect to A by the totalization formula

$$\hat{L}_A X \simeq \tau_{\geq 1} \lim_{\Delta} (\Omega^\infty A \otimes \Sigma^\infty)^{\bullet+1}(X).$$

1.2.2. *Equivariant spaces.* Our second application is to genuine equivariant homotopy theory. For a finite group G, we denote by $\mathcal{S}\mathrm{pc}(\mathrm{B}G) \simeq \mathcal{P}_\Sigma(\mathrm{Fin}_G)$ the ∞-category of genuine G-spaces, and by $\mathcal{SH}(\mathrm{B}G)$ its stabilization at representation spheres. See Section 6.3 for more details.

Theorem B (Equivariant homotopy theory, Theorems 6.20, 6.23 and 6.25)**.** *Let G be a finite group, $A \in \mathrm{CAlg}(\mathcal{SH}(\mathrm{B}G)_{\geq 0})$, and $X \in \mathcal{S}\mathrm{pc}(\mathrm{B}G)_*$ a nilpotent and $(1, G)$-connective (see Definition 6.9) G-space. Then, depending on $\underline{\pi}_0(A)$ we can identify $\hat{L}_A X$ as:*

$\underline{\pi}_0(A)$	Additional assumptions	Localization functor $\hat{L}_A$
$S^{-1}\underline{\pi}_0(\mathbb{1})$		$L_S X$
$\underline{\pi}_0(\mathbb{1})/n$		$L_n X$
$S^{-1}\underline{\mathbb{Z}}$	$|G| \in S$	$i_* L_S i^* X$
$\underline{\mathbb{Z}}/p$	G a finite p-group	$L_p X$

In these cases, we can moreover compute the localization of X with respect to A by the totalization formula

$$\hat{L}_A X \simeq \lim_{\Delta} (\Omega^\infty A \otimes \Sigma^\infty)^{\bullet+1}(X).$$

1.2.3. *Motivic spaces.* Our final application is to motivic homotopy theory. We fix a perfect field k and denote by $\mathcal{S}\mathrm{pc}(k)$ the ∞-category of motivic spaces over k [Mor99, MV99]. Denote its stabilization at $\mathbb{P}^1$ by $\mathcal{SH}(k)$. See Section 6.4 for more details.

Theorem C (Motivic homotopy theory, Theorems 6.39, 6.41, 6.48 and 6.51)**.** *Let k be a field, $A \in \mathrm{CAlg}(\mathcal{SH}(k)^{veff})$, and $X \in \mathcal{S}\mathrm{pc}(k)_*$ a resolvable motivic space (see Definition 6.32, e.g. any nilpotent 2-effective motivic space, cf. Remark 6.33). Let L be given as follows (depending on $\underline{\pi}_0(A)_*$):*

$\underline{\pi}_0(A)$	Additional assumptions	Localization functor L
$S^{-1}\underline{\pi}_0(\mathbb{1})_*$		L_S
$\underline{\pi}_0(\mathbb{1})_*/n$		L_n
$S^{-1}(\underline{\pi}_0(\mathbb{1})_*/\eta)$	$\mathrm{cd}_2(k) < \infty$	L_S
$\underline{\pi}_0(\mathbb{1})_*/(n, \eta)$	$\mathrm{cd}_2(k) < \infty$	L_n

Then in these cases we can identify the Bousfield–Kan completion of X as

$$LX \simeq \tau_{\geq 1} \lim_{\Delta} (\Omega^\infty A \otimes \Sigma^\infty)^{\bullet+1}(X).$$

Moreover, in the two cases where $L = L_S$, also $LX \simeq \hat{L}_A X$ (for X resolvable).

Remark 1.2. Contrary to the situations in Theorems A and B or when $L = L_S$, in the completion situation, i.e., $L = L_n$, we are unable to identify the homological localization $\hat{L}_A$. The issue is that, in contrast with L_n, it is not clear that if X is $\hat{L}_A$-local, then also $\tau_{\geq 1}X$ is. In other words, in contrast with the usual completion, it is not clear that discrete objects are $\hat{\hat{L}}_A$-local. See also [Mat24b, Conjecture 5.24] for further discussion.

1.3. **Application: unstable Adams spectral sequences.** The cosimplical object $\mathrm{CB}^\bullet$ obtained by the monadic resolution functor yields an unstable spectral sequence, which is a generalization of the classical unstable Adams spectral sequence, internal to the ∞-topos $\mathcal{X}$.

Proposition (Déglise–Pawar)**.** *Let* $L\colon \mathcal{X}_* \rightleftarrows \mathcal{D} \colon R$ *be an adjunction and* $X \in \mathcal{X}_*$. *Then for* $F_n = \mathrm{fib}(\mathrm{Tot}^n\, \mathrm{CB}^\bullet\, X \to \mathrm{Tot}^{n-1}\, \mathrm{CB}^\bullet\, X)$, *there is an unstable spectral sequence*

$$E_1^{r,s} = \underline{\pi}_{s-r}(F_r) \Rightarrow \underline{\pi}_{s-r}(\mathrm{Tot}\, \mathrm{CB}^\bullet\, X).$$

Proof. This follows from [DP24, Examples 2.32 and 2.37] for the tower $(\mathrm{Tot}^n\, \mathrm{CB}^\bullet)_n$. $\qquad\square$

Since we were able to identify the object $\mathrm{Tot}\, \mathrm{CB}^\bullet\, X$ in many cases, this spectral sequence takes a more digestible form, as recorded in the following theorem.

Theorem D. *The following unstable spectral sequences exist:*

(1) *Let* $\mathcal{X}$ *be an* ∞-*topos admitting a locally finite-dimensional cover,* $A \in \mathrm{CAlg}(\mathrm{Sp}(\mathcal{X})_{\geq 0})$ *and* $X \in \mathcal{X}_*$ *be nilpotent.*

$\underline{\pi}_0(A)$	Spectral Sequence
$S^{-1}\underline{\mathbb{Z}} \subset \underline{\mathbb{Q}}$	$E_1^{r,s} = \underline{\pi}_{s-r}(F_r) \Rightarrow \underline{\pi}_{s-r}(L_S X)$
$\underline{\mathbb{Z}}/n$	$E_1^{r,s} = \underline{\pi}_{s-r}(F_r) \Rightarrow \underline{\pi}_{s-r}(L_n X)$

(2) *Let* G *be a finite group,* $A \in \mathrm{CAlg}(\mathcal{SH}(BG)_{\geq 0})$, $X \in \mathcal{S}\mathrm{pc}(BG)_*$ *be nilpotent and* $(1, G)$-*connective.*

$\underline{\pi}_0(A)$	Additional assumptions	Spectral Sequence
$S^{-1}\underline{\pi}_0(\mathbb{1})$		$E_1^{r,s} = \underline{\pi}_{s-r}(F_r) \Rightarrow \underline{\pi}_{s-r}(L_S X)$
$\underline{\pi}_0(\mathbb{1})/n$		$E_1^{r,s} = \underline{\pi}_{s-r}(F_r) \Rightarrow \underline{\pi}_{s-r}(L_n X)$
$S^{-1}\underline{\mathbb{Z}}$	$|G| \in S$	$E_1^{r,s} = \underline{\pi}_{s-r}(F_r) \Rightarrow \underline{\pi}_{s-r}(i_* L_S i^* X)$
$\underline{\mathbb{Z}}/p$	G a finite p-group	$E_1^{r,s} = \underline{\pi}_{s-r}(F_r) \Rightarrow \underline{\pi}_{s-r}(L_p X)$

(3) *Let* k *be a field,* $A \in \mathrm{CAlg}(\mathcal{SH}(k)^{veff})$ *and* $X \in \mathcal{S}\mathrm{pc}(k)_*$ *be resolvable.*

$\underline{\pi}_0(A)$	Additional assumptions	Spectral Sequence
$S^{-1}\underline{\pi}_0(\mathbb{1})_*$		$E_1^{r,s} = \underline{\pi}_{s-r}(F_r) \Rightarrow \underline{\pi}_{s-r}(L_S X)$
$\underline{\pi}_0(\mathbb{1})_*/n$		$E_1^{r,s} = \underline{\pi}_{s-r}(F_r) \Rightarrow \underline{\pi}_{s-r}(L_n X)$
$S^{-1}(\underline{\pi}_0(\mathbb{1})_*/\eta)$	$\mathrm{cd}_2(k) < \infty$	$E_1^{r,s} = \underline{\pi}_{s-r}(F_r) \Rightarrow \underline{\pi}_{s-r}(L_S X)$
$\underline{\pi}_0(\mathbb{1})_*/(n,\eta)$	$\mathrm{cd}_2(k) < \infty$	$E_1^{r,s} = \underline{\pi}_{s-r}(F_r) \Rightarrow \underline{\pi}_{s-r}(L_n X)$

1.4. **Ingredients of the proof.** We consider a fairly general adjunction

$$L\colon \mathcal{X}_* \rightleftarrows \mathcal{D} \colon R.$$

To make progress, we shall assume that $\mathcal{X}$ is an ∞-topos. The case of sheaves is immediately applicable, as is the equivariant case because genuine G-spaces form a presheaf ∞-topos. With slightly more effort, using that motivic spaces are constructed as a localization of an ∞-topos, these will also fall into this setup.

In order to understand the Bousfield–Kan completion $\lim_{\Delta}(RL)^{\bullet+1}X$, we need to look at a more refined object. Namely, we want to consider the tower of partial totalizations as a *pro-object*

$$T_A^\bullet(X) = \lim_{\Delta^{\leq \bullet}} c(RL)^{\bullet+1}(X) \in \mathrm{Pro}(\mathcal{X}).$$

As pioneered in work of Bousfield, Farjoun and others (see e.g. [Dro73, Sul74, Gro75, EH76, AM86, Bou87]), pro-spaces behave like spaces in many ways, and similarly the ∞-category $\mathrm{Pro}(\mathcal{X})$ behaves in many ways like an ∞-topos. It turns out that the category behaves even more like an ∞-topos when considering a localization, namely $\mathrm{Pro}(\mathcal{X}_{<\infty})$. Here $\mathcal{X}_{<\infty} \subset \mathcal{X}$ is the subcategory of truncated objects.

Example 1.3. One way in which $\mathrm{Pro}(\mathcal{X}_{<\infty})$ behaves more like an ∞-topos than $\mathrm{Pro}(\mathcal{X})$ is that in the former ∞-category, geometric realizations are universal (i.e. stable under pullback), whereas in the latter they are (in general) not.

Remark 1.4. One advantage of working with pro-objects over totalizations is that the implicit infinite limit in a totalization does not interact well with connectivity in the case of ∞-topoi of positive homotopy dimension. In fact, in the main example we have in mind (i.e. motivic spaces), the underlying ∞-topos has infinite homotopy dimension, and so even sequential inverse limits can drop connectivity arbitrarily.

Another advantage is that one can prove that the pro-object $T_A^\bullet$ is compatible with geometric realizations (and thus principal fibrations) under relatively weak assumptions.

These properties will be made precise in Section 2. As we will see below, a major point will be the study of group actions in $\mathrm{Pro}(\mathcal{X}_{<\infty})$. The key takeaway is the comparison of $\mathrm{Pro}(\mathrm{Grp}(\mathcal{X}_{<\infty}))$ and $\mathrm{Grp}(\mathrm{Pro}(\mathcal{X}_{<\infty}))$ which are *not* equivalent, see Example 2.14. Nonetheless, the canonical comparison functor $\mathrm{Pro}(\mathrm{Grp}(\mathcal{X}_{<\infty})) \to \mathrm{Grp}(\mathrm{Pro}(\mathcal{X}_{<\infty}))$ is fully faithful, and if $G \in \mathrm{Grp}(\mathrm{Pro}(\mathcal{X}_{<\infty}))$ is pro-connected, then, surprisingly, it lies in the essential image of this functor, see Theorem 2.15. For those groups in the essential image (which we will call *levelwise groups*), one can lift the ∞-bundle theory of [NSS14] to the ∞-category of pro-truncated objects. Namely, we obtain the following theorem:

Theorem E (Propositions 2.11 and 2.17 and Example 2.18)**.** *Let $G \in \mathrm{Grp}(\mathrm{Pro}(\mathcal{X}_{<\infty}))$ be a levelwise group. Then $\Omega \mathrm{B}G \simeq G$. Moreover, there is a canonical equivalence between maps $f\colon X \to \mathrm{B}G$ and objects F with a G-action, given by sending f to $\mathrm{fib}(f)$ with the induced action by $G = \Omega \mathrm{B}G$. If F is an object with a G-action, we get a fiber sequence*

$$F \to F /\!\!/ G \to \mathrm{B}G.$$

Once this machinery is set up, we proceed to investigate the functor $\tau_{<\infty} T_A^\bullet$ (i.e. the reflection of $T_A^\bullet$ into $\mathrm{Pro}(\mathcal{X}_{<\infty})$). Our main result about it is a version of Bousfield and Kan's principal fibration lemma, whose proof will concern the first half of this article:

Theorem F (Theorem 5.7)**.** *The functor $\tau_{<\infty} T_A^\bullet \colon \mathcal{X}_* \to \mathrm{Pro}(\mathcal{X}_{*,<\infty})$ preserves those fiber sequences $F \to E \to B$ where B is simply connected and F is connected, and $LF, L\Omega B$ satisfy certain connectivity assumptions (see Section 3 for more details).*

This theorem will allow us to induct on the principalized Postnikov tower of any good enough nilpotent sheaves in $\mathcal{X}_*$, thus essentially reducing the convergence question to the case of an infinite loop sheaf.

1.5. **Organization.** In Section 2, we establish properties of pro-truncated objects in ∞-topoi alluded to above, in particular the principal fibration theory. Then in Section 3 we introduce notation and common assumptions for the rest of the article, most importantly, the adjunction $\mathcal{X} \rightleftarrows \mathcal{D}$ and the axioms (C), (M) and (S). Section 4 investigates compatibility of the localization functor $\hat{L}$ with fiber sequences. In a sense our main results are contained in Section 5: here we prove that, under a certain fairly general list of assumptions, the Bousfield–Kan completion functor $\tau_{<\infty} \mathrm{Tot}^\bullet \mathrm{CB}$ preserves principal fibrations. We then come to Section 6, where we establish our applications to sheaves, equivariant spaces and motivic spaces. We conclude with some appendices: in Section A we recall how to extract (co)simplicial data out of adjunctions, in Section B we prove a well-known fact about the stable analog of Bousfield–Kan completion, in Section C we extend the unstable p-completion theory of [Mat24b] to unstable n-completion (n not necessarily prime), and in Section D we establish a minor extension of Levine's convergence theorem in motivic homotopy theory.

1.6. **Notation and conventions.** We freely use the language of ∞-categories as set out in [Lur09, Lur17]. A *space* will be an object of the ∞-category $\mathcal{S}\mathrm{pc}$ of spaces/homotopy types/∞-groupoids/anima, and a *sheaf* will be an object of an ∞-topos. Here is a table of some of the notation that we use.

notation	meaning	reference/definition
$\mathrm{Pro}(\mathcal{C})$	∞-category of pro-objects	Section 2.1
c	constant pro-object functor	Section 2.1
$\mathrm{mat}\, X$	materialization of a pro-object	Section 2.1
$\mathcal{X}_{<\infty}$	truncated objects	Section 2.2
$\tau_{<\infty}$	pro-truncation	Section 2.2
$\mathrm{Grpd}(\mathcal{C})$	∞-category of groupoid objects	Section 2.4
$\mathrm{Grp}(\mathcal{C})$	∞-category of group objects	Section 2.4
$L, R, \mathcal{D}^{\mathrm{sld}}$	standard setup	Section 3
$\hat{L}$	homological localization functor	Section 4
$\mathrm{CB}^\bullet$	cobar resolution	Section 5.1
$T^\bullet$	monadic resolution tower	Section 5.1

C^n	total fiber of the codegeneracy cube	Section 5.3
S^n	total fiber of the face cube	Section 5.3
L_n	unstable n-completion functor	[Mat24b, §3], Section C
L_S	unstable S-localization functor	[Mat24a, §3]
$\mathcal{S}\mathrm{pc}(\mathrm{B}G)$	genuine G-spaces	$\mathcal{P}_\Sigma(\mathrm{Fin}_G)$
$\mathcal{SH}^{S^1}(\mathrm{B}G)$	G-equivariant S^1-spectra	$\mathrm{Sp}(\mathcal{S}\mathrm{pc}(\mathrm{B}G))$
$\mathcal{SH}(\mathrm{B}G)$	genuine G-spectra	Section 6.3
$\underline{\pi}_i E$	Mackey functor valued homotopy objects	Section 6.3
$\underline{A}$	Burnside ring Mackey functor	e.g. [LMSM86, §V.9]
Sm_k	smooth quasi-compact k-schemes	
$\mathcal{S}\mathrm{pc}(k)$	motivic spaces	Section 6.4
$\mathcal{SH}^{S^1}(k)$	motivic S^1-spectra	$\mathrm{Sp}(\mathcal{S}\mathrm{pc}(k))$
$\mathcal{SH}(k)$	motivic spectra	Section 6.4
$\mathcal{S}\mathrm{pc}(k)_{\geq(\nu,\nu)}$	ν-effective motivic spaces	[ABH24, §3], Remark 6.26
$\underline{\pi}_i(X)_*$	stable motivic homotopy sheaves	Remark 6.27
$\mathcal{SH}(k)^\heartsuit$	heart of the homotopy t-structure	Remark 6.27
$\mathcal{SH}(k)^{\mathrm{veff}}$	very effective motivic spectra	Remark 6.27
$\underline{\pi}_i^{\mathrm{eff}}(X)$	stable effective motivic homotopy sheaves	Remark 6.27
$\mathcal{SH}(k)^{\mathrm{eff}\heartsuit}$	heart of the effective t-structure	Remark 6.27
f_n, s_n	effective cover and slice functors	Remark 6.28
$\underline{K}^{MW}$	Milnor–Witt K-theory homotopy module	
$\underline{K}^M$	Milnor K-theory homotopy module	
η	motivic Hopf map	

2. Homotopy theory in pro-topoi

Throughout we let $\mathcal{X}$ be an ∞-topos [Lur09, Definition 6.1.0.4]. In this section we will show that the ∞-category $\mathrm{Pro}(\mathcal{X}_{<\infty})$ of pro-truncated objects in $\mathcal{X}$ behaves in some ways like an ∞-topos. In particular, we will study the relationship between groupoids and (some kind of) epimorphisms in $\mathrm{Pro}(\mathcal{X}_{<\infty})$ (see Proposition 2.11), and between group actions and maps to classifying spaces (see Proposition 2.17). This culminates in our replacement for principal fibration theory in $\mathrm{Pro}(\mathcal{X}_{<\infty})$: for a pro-connected group $G \in \mathrm{Grp}(\mathrm{Pro}(\mathcal{X}_{<\infty}))$ there is an equivalence between G-torsors and maps to BG, given as usual by taking G-orbits respectively taking fibers (see Example 2.18 and Theorem 2.15).

2.1. **Pro-objects.** Given a possibly large ∞-category $\mathcal{C}$, we denote by $\mathrm{Pro}(\mathcal{C})$ the ∞-category of pro-objects in $\mathcal{C}$ [BGH18, §0.11.6]. One possible definition is $\mathrm{Pro}(\mathcal{C}) = \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. This comes with the *constant object* functor

$$c\colon \mathcal{C} \to \mathrm{Pro}(\mathcal{C}),$$

characterizing $\mathrm{Pro}(\mathcal{C})$ via a universal property: $\mathrm{Pro}(\mathcal{C})$ has cofiltered limits and any functor out of $\mathcal{C}$ into an ∞-category with cofiltered limits extends uniquely along c to a functor out of $\mathrm{Pro}(\mathcal{C})$ preserving cofiltered limits. We note the following further facts:

- c preserves finite limits (see the dual of [Lur09, Proposition 5.3.5.14]).
- The association $\mathcal{C} \mapsto \mathrm{Pro}(\mathcal{C})$ canonically assembles into a functor

$$\mathrm{Pro}(-)\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty.$$

- Any object in $\mathrm{Pro}(\mathcal{C})$ can be presented as a cofiltered limit of constant objects.
- c is fully faithful.
- If $\mathcal{C}$ has cofiltered limits, then c has a right adjoint $\mathrm{mat}\colon \mathrm{Pro}(\mathcal{C}) \to \mathcal{C}$, given by $\mathrm{mat}\lim_i cX_i \simeq \lim_i X_i$.

From this we can deduce that there is an equivalence $\mathrm{Fun}(I, \mathrm{Pro}(\mathcal{C})) \simeq \mathrm{Pro}(\mathrm{Fun}(I, \mathcal{C}))$ for any finite poset I [Lur09, Proposition 5.3.5.15]. It follows, for example, that $\mathrm{Pro}(\mathcal{C})_* \simeq \mathrm{Pro}(\mathcal{C}_*)$. However, this fails for infinite diagrams: in general, the canonical functor

$$\mathrm{Pro}(\mathrm{Fun}(I, \mathcal{C})) \to \mathrm{Fun}(I, \mathrm{Pro}(\mathcal{C}))$$

is neither fully faithful nor essentially surjective. We will revisit this theme several times in what follows.

2.2. **Pro-truncated objects.** We write $\mathcal{X}_{<\infty} \subset \mathcal{X}$ for the subcategory of truncated objects, and $\mathrm{Pro}(\mathcal{X}_{<\infty}) \subset \mathrm{Pro}(\mathcal{X})$ for the subcategory of pro-truncated objects. The latter inclusion has a left adjoint

$$\tau_{<\infty} = \lim_n \mathrm{Pro}(\tau_{\leq n}) \colon \mathrm{Pro}(\mathcal{X}) \to \mathrm{Pro}(\mathcal{X}_{<\infty})$$

called protruncation; see [BGH18, §4.1.2]. It is pro-induced by the functor $\mathcal{X} \to \mathrm{Pro}(\mathcal{X}_{<\infty})$ given by $X \mapsto \lim_n c\tau_{\leq n} X$. It preserves all limits [HHW22, Proposition 3.9] (the reference only proves it for the ∞-topos of spaces, but the proof works for an arbitrary ∞-topos).

Recall the notion of almost finite colimits from [Hai24, Definition 1.9] (and in particular the notion of n-colimit-cofinal functors, cf. [Hai24, Definition 1.7]). Let us give an example of almost finite colimits:

Lemma 2.1. *Geometric realizations of both simplicial and semisimplicial objects are almost finite.*

Proof. We have to show that both Δ^{op} and $\Delta^{\mathrm{inj,op}}$ have n-colimit-cofinal approximations by a finite category. Since the inclusion $\Delta^{\mathrm{inj,op}} \hookrightarrow \Delta^{\mathrm{op}}$ is colimit-cofinal (see [Lur09, Lemma 6.5.3.7]), it suffices to prove this for $\Delta^{\mathrm{inj,op}}$. For any n consider the functor $\Delta^{\mathrm{inj,op}}_{\leq n} \to \Delta^{\mathrm{inj,op}}$. It is clear that $\Delta^{\mathrm{inj,op}}_{\leq n}$ is a finite category, and the functor is n-colimit-cofinal by [Du23, Example 6.14]. $\square$

The first part of the next proposition was proven by Haine for the ∞-category of spaces, cf. [Hai24, Proposition 1.17], and is the main motivation for considering the notion of almost finite colimits.

Proposition 2.2 (Haine)**.** *Almost finite colimits in $\mathrm{Pro}(\mathcal{X}_{<\infty})$ are universal and commute with cofiltered limits. In particular geometric realizations (of either simplicial or semisimplicial objects) in $\mathrm{Pro}(\mathcal{X}_{<\infty})$ are universal and commute with cofiltered limits.*

In order to prove this, we will use two further properties of $\mathrm{Pro}(\tau_{\leq n})$.

Lemma 2.3. *Let $f \colon X \to Y$ be a morphism in $\mathrm{Pro}(\mathcal{X})$. Then $\tau_{<\infty} f$ is an equivalence if and only if $\mathrm{Pro}(\tau_{\leq n})(f)$ is an equivalence for all n.*

Proof. This is immediate from the facts that $\tau_{<\infty} \simeq \lim_n \mathrm{Pro}(\tau_{\leq n})$ and $\mathrm{Pro}(\tau_{\leq n}) \simeq \mathrm{Pro}(\tau_{\leq n}) \circ \tau_{<\infty}$. (The latter holds because $\mathrm{Pro}(\mathcal{X}_{\leq n}) \subset \mathrm{Pro}(\mathcal{X}_{<\infty}) \subset \mathrm{Pro}(\mathcal{X})$ and the truncations are left adjoints to the appropriate inclusions.) $\square$

The following lemma is an extension of [DH21, Proposition 4.13] to pro-objects in ∞-topoi.

Lemma 2.4. *Let $\mathcal{X}$ be an ∞-topos. The pullback functor*

$$\mathrm{Fun}(\Lambda_2^2, \mathrm{Pro}(\mathcal{X})) \to \mathrm{Pro}(\mathcal{X})$$

sends $\mathrm{Pro}(\tau_{\leq n+1})$-equivalences to $\mathrm{Pro}(\tau_{\leq n})$-equivalences.

Proof. By 2-out-of-3, it suffices to show that the n-truncated pullback functor inverts the map of spans $(U \to V \leftarrow W) \to (\mathrm{Pro}(\tau_{\leq n+1})(U) \to \mathrm{Pro}(\tau_{\leq n+1})(V) \leftarrow \mathrm{Pro}(\tau_{\leq n+1})(W))$. Since all functors in sight preserve cofiltered limits, this reduces to the case where U, V, W are constant, which is a special case of [DH21, Proposition 4.13]. $\square$

Also note the following general fact about categories of pro-objects:

Proposition 2.5. *Let $\mathcal{C}$ be any ∞-category with finite colimits. Then finite colimits commute with cofiltered limits in $\mathrm{Pro}(\mathcal{C})$.*

Proof. Mapping to objects cX with $X \in \mathcal{C}$ is jointly conservative, and turns cofiltered limits to filtered colimits (and finite colimits to finite limits). Hence, the claim follows from the corresponding dual result about spaces, cf. [Lur09, Proposition 5.3.3.3]. $\square$

Proof of Proposition 2.2. We first show that almost finite colimits commute with cofiltered limits in $\mathrm{Pro}(\mathcal{X}_{<\infty})$, which is slightly easier. Using Lemma 2.3 and the fact that $\mathrm{Pro}(\tau_{\leq n})$ preserves colimits (being a left adjoint) and cofiltered limits, it suffices to prove the same statement about $\mathrm{Pro}(\mathcal{X}_{\leq n})$. But almost finite colimits in $\mathrm{Pro}(\mathcal{X}_{\leq n})$ are just finite colimits, and finite colimits always commute with cofiltered limits in categories of pro-objects by Proposition 2.5.

Now we show that almost finite colimits are universal in $\mathrm{Pro}(\mathcal{X}_{<\infty})$. Using Lemmas 2.3 and 2.4 we reduce to proving the same statement for $\mathrm{Pro}(\mathcal{X}_{\leq n+1})$. But almost finite colimits in this category are finite colimits, so the result holds by [Hai24, Lemma 1.1]. $\square$

2.3. Epimorphisms and connected pro-objects.
Write $\mathrm{Fun}(\Delta, \mathcal{X})^{\mathrm{epi}} \subset \mathrm{Fun}(\Delta, \mathcal{X})$ for the full subcategory spanned by those morphisms that are effective epimorphisms [Lur09, the discussion after Corollary 6.2.3.5].

Definition 2.6. We say that $f \in \mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{X}))$ is a *levelwise effective epimorphism* if under the canonical equivalence $\mathrm{Pro}(\mathrm{Fun}(\Delta^1, \mathcal{X})) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{X}))$ the morphism f corresponds to an object in the full subcategory $\mathrm{Pro}(\mathrm{Fun}(\Delta^1, \mathcal{X})^{\mathrm{epi}})$.

We write $\mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{X}))^{\mathrm{epi}} \subset \mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{X}))$ for the full subcategory of levelwise effective epimorphisms, so that there is a canonical equivalence $\mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{X}))^{\mathrm{epi}} \simeq \mathrm{Pro}(\mathrm{Fun}(\Delta^1, \mathcal{X})^{\mathrm{epi}})$.

We say that $f \in \mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{X}_{<\infty}))$ is a *levelwise effective epimorphism* if it is one considered as an object of $\mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{X}))$, i.e., if it can be presented as a cofiltered limit of effective epimorphisms between truncated objects (use that $\tau_{\leq n}$ preserves effective epimorphisms and hence $\tau_{<\infty}$ preserves levelwise effective epimorphisms).

Remark 2.7. If $f \colon X \to Y$ is a morphism in $\mathcal{X}$, then f is an effective epimorphism if and only if $\pi_0(f)$ is surjective (as a map of sheaves, i.e., an epimorphism in the 1-category $\mathcal{X}_{\leq 0}$), essentially by [Lur09, Corollary 6.2.3.5]. In particular, we see that $g \colon U \to V$ in $\mathrm{Pro}(\mathcal{X})$ is a levelwise effective epimorphism if and only if $\mathrm{Pro}(\pi_0)(g)$ is a levelwise surjection.

Definition 2.8. We say that an object $X \in \mathrm{Pro}(\mathcal{X})_*$ is *pro-connected* if the map $* \to X$ is a levelwise effective epimorphism. We write $\mathrm{Pro}(\mathcal{X})_{*,\geq 1} \subset \mathrm{Pro}(\mathcal{X})_*$ for the full subcategory of pro-connected objects.

Lemma 2.9. *Let $X \in \mathrm{Pro}(\mathcal{X})_*$ be an object. X is pro-connected if and only if X lives in the full subcategory $\mathrm{Pro}(\mathcal{X}_{*,\geq 1}) \subseteq \mathrm{Pro}(\mathcal{X}_*) \simeq \mathrm{Pro}(\mathcal{X})_*$.*

Proof. Suppose first that $X \in \mathrm{Pro}(\mathcal{X}_{*,\geq 1})$. Then there exists a cofiltered limit presentation $X = \lim_i cX_i$, with X_i connected. In particular, $* \to X_i$ is an effective epimorphism for all i, which shows that $* \to X$ is a levelwise effective epimorphism.

Suppose on the other hand that $* \to X$ is a levelwise effective epimorphism. Thus, this map can be written as a cofiltered limit of effective epimorphisms $A_i \to X_i$ in $\mathcal{X}_*$. Set $X_i' = X_i \amalg_{A_i} *$. Then since c preserves colimits and finite colimits commute with cofiltered limits in pro-objects by Proposition 2.5, we find
$$\lim_i cX_i' \simeq \lim_i cX_i \amalg_{\lim_i cA_i} * \simeq \lim_i cX_i \simeq X.$$
(Indeed $\lim_i cA_i = *$ by assumption.) By construction X_i' is connected (as effective epimorphisms are stable under cobasechange, cf. [Lur09, Corollary 6.5.1.17 with $n = 0$]), concluding the proof. $\square$

2.4. Groupoids.
Let $\mathcal{C}$ be either $\mathcal{X}$, $\mathrm{Pro}(\mathcal{X})$, or $\mathrm{Pro}(\mathcal{X}_{<\infty})$, so that we can speak about the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})^{\mathrm{epi}}$, cf. Section 2.3. By a groupoid in $\mathcal{C}$ we mean an object of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ satisfying the Segal condition [Lur09, Proposition 6.1.2.6 (4")]. Left Kan extending to Δ_+^{op} and restricting to $\Delta_{+,\leq 0}^{\mathrm{op}} \simeq \Delta^1$ yields a functor $|-|$, participating in an adjunction
$$|-| \colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}) \rightleftarrows \mathrm{Fun}(\Delta^1, \mathcal{C}) \colon N.$$
By construction, the right adjoint is given by right Kan extending to Δ_+^{op} and then restricting to Δ^{op}. It is clear that the image of N is contained in $\mathrm{Grpd}(\mathcal{C})$. If $\mathcal{C} = \mathcal{X}$, then the image of $|-|$ consists of effective epimorphisms, cf. [Lur09, Lemma 6.2.3.13] (using that each of the morphisms $X_n \to |X_\bullet|$ factors over X_0).

Lemma 2.10. *If $\mathcal{C} = \mathrm{Pro}(\mathcal{X})$ or $\mathcal{C} = \mathrm{Pro}(\mathcal{X}_{<\infty})$, then the image of $|-|$ consists of levelwise effective epimorphisms.*

Proof. By Remark 2.7, we have to see that for any $X_\bullet \in \mathrm{Grpd}(\mathcal{C})$ the map $\mathrm{Pro}(\pi_0)(X_0) \to \mathrm{Pro}(\pi_0)(|X_\bullet|)$ is a levelwise surjection. Now, $\mathrm{Pro}(\pi_0)(|X_\bullet|) \simeq \mathrm{Pro}(\pi_0)(\mathrm{colim}_{\Delta_{\leq 1}^{\mathrm{inj,op}}} X_\bullet)$ by [Lur17, Lemma 1.2.4.17] and [HP23, Proposition A.1] (since $\mathrm{Pro}(\pi_0)$ commutes with the colimit, and $\mathrm{Pro}(\mathcal{X}_{\leq 0})$ is a 1-category), and hence the question depends only on the finite diagram $X_\bullet|_{\Delta_{\leq 1}^{\mathrm{inj,op}}}$. The result now follows using the equivalence $\mathrm{Pro}(\mathrm{Fun}(\Delta_{\leq 1}^{\mathrm{inj,op}}, \mathcal{X})) \simeq \mathrm{Fun}(\Delta_{\leq 1}^{\mathrm{inj,op}}, \mathrm{Pro}(\mathcal{X}))$, and by using [Lur09, Lemma 6.2.3.13] on each level. $\square$

We write $\mathrm{Grp}(\mathcal{C}) \subset \mathrm{Grpd}(\mathcal{C})$ for the full subcategory of *group objects*, i.e. those $U_\bullet$ such that $U_0 = *$.

Proposition 2.11. *(1) We have an equivalence* $\mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{X}_{<\infty}))^{\mathrm{epi}} \simeq \mathrm{Pro}(\mathrm{Grpd}(\mathcal{X}_{<\infty}))$, *and thus* $\mathrm{Pro}(\mathcal{X}_{<\infty})_{*,\geq 1} \simeq \mathrm{Pro}(\mathrm{Grp}(\mathcal{X}_{<\infty}))$.

 (2) The composite

$$\mathrm{Pro}(\mathrm{Grpd}(\mathcal{X}_{<\infty})) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{X}_{<\infty}))^{\mathrm{epi}} \xrightarrow{N} \mathrm{Grpd}(\mathrm{Pro}(\mathcal{X}_{<\infty}))$$

 is the canonical functor.
 (3) The functor $N\colon \mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{X}_{<\infty}))^{\mathrm{epi}} \to \mathrm{Grpd}(\mathrm{Pro}(\mathcal{X}_{<\infty}))$ *is fully faithful.*
 (In fact the first two statements hold for all of $\mathrm{Pro}(\mathcal{X})$*.)*

Proof. (1) The fact that $\mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{X}))^{\mathrm{epi}} \simeq \mathrm{Pro}(\mathrm{Grpd}(\mathcal{X}))$ follows immediately since by definition $\mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{X}))^{\mathrm{epi}} \simeq \mathrm{Pro}(\mathrm{Fun}(\Delta^1, \mathcal{X})^{\mathrm{epi}})$ and $\mathrm{Fun}(\Delta^1, \mathcal{X})^{\mathrm{epi}} \simeq \mathrm{Grpd}(\mathcal{X})$; for the latter see [Lur09, Theorem 6.1.0.6].

To get the result for pro-truncated objects, it will suffice to show that the equivalence $\mathrm{Fun}(\Delta^1, \mathcal{X})^{\mathrm{epi}} \simeq \mathrm{Grpd}(\mathcal{X})$ restricts to an equivalence $\mathrm{Fun}(\Delta^1, \mathcal{X}_{<\infty})^{\mathrm{epi}} \simeq \mathrm{Grpd}(\mathcal{X}_{<\infty})$. Let us first assume that $f\colon X \to Y$ is an effective epimorphism, such that both X and Y are in $\mathcal{X}_{<\infty}$, say in $\mathcal{X}_{\leq M}$ for some $M \gg 0$. Now, $N(f)$ is (the restriction of) the right Kan extension of f to Δ_+. As right Kan extensions are computed using limits, and $\mathcal{X}_{\leq M} \subset \mathcal{X}$ is stable under limits, we see that $N(f) \in \mathrm{Grpd}(\mathcal{X}_{\leq M}) \subseteq \mathrm{Grpd}(\mathcal{X}_{<\infty})$. For the other direction, assume that $X_\bullet \in \mathrm{Grpd}(\mathcal{X}_{<\infty})$. Pulling back $X_0 \to |X_\bullet|$ along itself yields $X_1 \to X_0$ (since $N(|X_\bullet|) \simeq X_\bullet$). This map being truncated [Lur09, Lemma 5.5.6.14], we deduce that $X_0 \to |X_\bullet|$ is also truncated [Lur09, Proposition 6.2.3.17]. Thus, $|X_\bullet|$ is truncated by Lemma 2.12 below.

That this equivalence restricts further to an equivalence $\mathrm{Pro}(\mathcal{X}_{<\infty})_{*,\geq 1} \simeq \mathrm{Pro}(\mathrm{Grp}(\mathcal{X}_{<\infty}))$ is clear.

(2) Holds by construction.

(3) We want to show that $|N(-)| \simeq \mathrm{id}$. Since geometric realizations commute with cofiltered limits (Proposition 2.2), so does the functor $|-|$ (and so does N, being a right adjoint). Any object in $\mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{X}_{<\infty}))^{\mathrm{epi}}$ being a cofiltered limit of effective epimorphisms in $\mathcal{X}_{<\infty}$, we reduce to such (constant) objects. This case follows from the same result for $\mathcal{X}$, for which see [Lur09, Theorem 6.1.0.6] (where N is even an equivalence). □

We used the following fact, for which we could not locate a reference.

Lemma 2.12. *Let $\mathcal{X}$ be an ∞-topos and $f\colon X \to Y \in \mathcal{X}$ an n-truncated effective epimorphism. Then X is $(n+1)$-truncated if and only if Y is $(n+1)$-truncated.*

Proof. If Y is $(n+1)$-truncated then so is X by [Lur09, Lemma 5.5.6.14]. We establish the converse. Let $L\colon \mathcal{P}(\mathcal{C}) \rightleftarrows \mathcal{X} \colon R$ be a presentation of $\mathcal{X}$ as a left exact localization of a presheaf ∞-topos [Lur09, Definition 6.1.0.4]. Recall that both R and L preserve i-truncated morphisms and objects [Lur09, Proposition 5.5.6.16]. Write $RX \xrightarrow{f_0} Y_0 \to RY$ for the epi-mono factorization [Lur17, Example 5.2.8.16 for $n = -1$] of Rf. Since Rf is n-truncated and $Y_0 \to RY$ is (-1)-truncated (whence n-truncated), $RX \to Y_0$ is also n-truncated (apply [Lur09, Proposition 5.5.6.16] in $\mathcal{P}(\mathcal{C})_{/RY}$). As $X \to Y$ is an effective epimorphism and L preserves epi-mono factorizations, we see that $Lf_0 \simeq f$. Using once more that L preserves i-truncated maps, we see that we may replace $\mathcal{X}$ by $\mathcal{P}(\mathcal{C})$ and thus reduce to $\mathcal{X} = \mathcal{S}\mathrm{pc}$. In this case the claim follows by examination of the long exact sequence of homotopy groups. □

Thanks to the above result, the following makes good sense.

Definition 2.13. We call objects in the essential image of the fully faithful functor $\mathrm{Pro}(\mathrm{Grpd}(\mathcal{X}_{<\infty})) \to \mathrm{Grpd}(\mathrm{Pro}(\mathcal{X}_{<\infty}))$ *levelwise groupoids*. *Levelwise groups* are defined similarly.

Example 2.14. For any group $G \in \mathrm{Grp}(\mathrm{Pro}(\mathcal{X}_{<\infty}))$, Proposition 2.11 supplies us with a map $G \to \Omega BG(\colon= N|G|)$, which is in fact the initial map to a levelwise group. However, not all groups are levelwise groups, as the following example shows. Consider the pro-set $G \coloneqq \lim_k cG_k$, where $G_k \coloneqq [-\frac{1}{2^k}, \frac{1}{2^k}] \subset \mathbb{R}$ is a closed interval (considered as a set), and the transition morphisms are given by inclusion. We define a group structure on G as the inverse limit of the maps $G_{k+1} \times G_{k+1} \to G_k$ sending (x, y) to $x + y$ (where the addition is the addition from $\mathbb{R}$). We claim that G is not a levelwise group. For this, consider the initial map to a levelwise group $G \to \Omega BG$; it suffices to show that this map is not an equivalence. This can be checked after materializing. The left-hand side yields $\mathrm{mat}(G) = \lim_k G_k = \cap_k G_k = \{0\}$. It thus suffices to show that $\mathrm{mat}(\Omega BG) = \Omega \mathrm{mat}(BG) \neq *$. In fact, $\pi_1(\mathrm{mat}(BG)) \neq 0$, as we shall show now. Denote by $B_{k,\bullet}$ the following inverse system of *semi*-simplicial diagrams:

$$B_{k,\bullet} = (* \Leftarrow G_k \Lleftarrow G_{k+1} \times G_{k+1} \Lleftarrow G_{k+2} \times G_{k+2} \times G_{k+2} \dots).$$

Then $|\lim_k cB_{k,\bullet}| \simeq \mathrm{B}G$. Using that in pro-truncated spaces cofiltered limits commute with geometric realizations (even of semisimplicial objects), and c is a fully faithful left adjoint (with right adjoint the materialization) we deduce that $\mathrm{mat}(\mathrm{B}G) \simeq \lim_k \mathrm{B}G_k$, where $\mathrm{B}G_k := |B_{k,\bullet}|$. Thus, π_1 of the materialization is nonzero if we can find a compatible family of nonzero elements in the $\pi_1(\mathrm{B}G_k)$. Indeed, such a family then defines a nonzero element of $\lim_k \pi_1(\mathrm{B}G_k)$, and there is a surjection $\pi_1(\lim_k \mathrm{B}G_k) \to \lim_k \pi_1(\mathrm{B}G_k)$. For every $x \in G_k$ we get an element $[x] \in \pi_1(\mathrm{B}G_k)$. There is a canonical map $\mathrm{B}G_k \to \mathrm{B}\mathbb{R}$ (where we view $\mathbb{R}$ as a discrete group) which maps the loop $[x]$ to the corresponding loop $[x] \in \pi_1(\mathrm{B}\mathbb{R})$, and thus those elements are all distinct (and in particular nonzero if x is nonzero). Now set $x_n = [\frac{1}{2^n}] \cdots [\frac{1}{2^n}] \in \pi_1(\mathrm{B}G_n)$, the 2^n-fold product (in π_1) of the loop represented by $\frac{1}{2^n}$. Since in $\mathrm{B}G_{n-1}$ we impose the relation $[\frac{1}{2^n}] \cdot [\frac{1}{2^n}] = [\frac{1}{2^{n-1}}]$, it follows that x_n maps to x_{n-1}. By the argument above, this compatible family is nonzero (as it gets mapped to $[1] \in \mathrm{B}\mathbb{R}$), thus proving the claim.

We have the following useful criterion for detecting levelwise groups.

Theorem 2.15. *Let $G \in \mathrm{Grp}(\mathrm{Pro}(\mathcal{X}_{<\infty}))$ be pro-connected, i.e., the unit $* \to G$ is a levelwise effective epimorphism. Then G is a levelwise group.*

For the proof, we need the following identification of the $\mathcal{E}_1$-structures on the two different bar constructions one can perform on an $\mathcal{E}_2$-group, for which we were unable to locate a proof in the literature. For any ∞-category $\mathcal{C}$ with finite products and geometric realizations, and any $G \in \mathrm{Grp}(\mathcal{C})$, write $\mathrm{B}_\bullet G \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ for the corresponding simplicial object, and $\mathrm{B}_\mathcal{C} G := \mathrm{B}G := |\mathrm{B}_\bullet G| \in \mathcal{C}$ for the geometric realization.

Proposition 2.16. *Let $\mathcal{C}$ be an ∞-category with finite products and all colimits, such that geometric realizations commute with finite products (e.g. $\mathcal{X}$ or $\mathrm{Pro}(\mathcal{X}_{<\infty})$). There is an equivalence of functors $\mathrm{Grp}(\mathrm{B}_\mathcal{C}) \simeq \mathrm{B}_{\mathrm{Grp}(\mathcal{C})} : \mathrm{Grp}(\mathrm{Grp}(\mathcal{C})) \to \mathrm{Grp}(\mathcal{C})$.*

Proof. Let $\mathcal{C}_0 \subset \mathcal{C}$ be a small full subcategory closed under finite products. Consider the commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Grp}(\mathrm{Grp}(\mathcal{C}_0)) & \xrightarrow{y} & \mathrm{Grp}(\mathrm{Grp}(\mathcal{P}(\mathcal{C}_0))) & \xrightarrow{\mathrm{colim}} & \mathrm{Grp}(\mathrm{Grp}(\mathcal{C})) \\
& & \Big\Downarrow & & \Big\Downarrow \\
& & \mathrm{Grp}(\mathcal{P}(\mathcal{C}_0)) & \xrightarrow{\mathrm{colim}} & \mathrm{Grp}(\mathcal{C}),
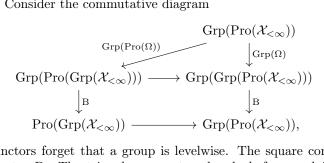\end{array}
$$

where $y : \mathcal{C}_0 \to \mathcal{P}(\mathcal{C}_0)$ and $\mathrm{colim} : \mathcal{P}(\mathcal{C}_0) \to \mathcal{C}$ preserve finite products (whence groups) and the vertical arrows are $\mathrm{Grp}(\mathrm{B})$ respectively $\mathrm{B}_{\mathrm{Grp}}$. An equivalence between the left hand vertical arrows thus yields an equivalence between the two composites $\mathrm{Grp}(\mathrm{Grp}(\mathcal{C}_0)) \to \mathrm{Grp}(\mathcal{C})$. Writing $\mathcal{C}$ as the filtered colimit of small full subcategories $\mathcal{C}_0$ closed under finite products, we can reduce from arbitrary $\mathcal{C}$ to $\mathcal{C} = \mathcal{P}(\mathcal{C}_0)$. In fact, since $\mathrm{Grp}(\mathcal{C}) \simeq \mathrm{Grp}(\mathcal{C}_*)$, we may reduce further to $\mathcal{C} = \mathcal{P}(\mathcal{C}_0)_*$. I.e., from now on we can (and will) assume that $\mathcal{C} = \mathcal{X}_*$ for an ∞-topos $\mathcal{X}$.

Note that $\mathrm{Grp}(\mathrm{Grp}(\mathcal{C})) \subset \mathrm{Fun}(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathcal{C})$ and under this inclusion, the different directions in which we can take geometric realizations on a bisimplicial object correspond to the two functors $\mathrm{Grp}(\mathrm{B}_\mathcal{C})$ and $\mathrm{B}_{\mathrm{Grp}(\mathcal{C})}$. In particular, since colimits commute, we see that

$$\mathrm{B} \circ \mathrm{Grp}(\mathrm{B}) \simeq \mathrm{B} \circ \mathrm{B}_{\mathrm{Grp}} : \mathrm{Grp}(\mathrm{Grp}(\mathcal{C})) \to \mathcal{C}.$$

The result follows since in an ∞-topos $\mathcal{X}$, the functor $\mathrm{B} : \mathrm{Grp}(\mathcal{X}_*) \simeq \mathrm{Grp}(\mathcal{X}) \to \mathcal{X}_*$ is fully faithful [Lur09, Lemma 7.2.2.11]. □

Proof of Theorem 2.15. Consider the commutative diagram

$$
\begin{array}{ccc}
 & & \mathrm{Grp}(\mathrm{Pro}(\mathcal{X}_{<\infty})) \\
 & \overset{\mathrm{Grp}(\mathrm{Pro}(\Omega))}{\swarrow} & \Big\downarrow {\scriptstyle \mathrm{Grp}(\Omega)} \\
\mathrm{Grp}(\mathrm{Pro}(\mathrm{Grp}(\mathcal{X}_{<\infty}))) & \longrightarrow & \mathrm{Grp}(\mathrm{Grp}(\mathrm{Pro}(\mathcal{X}_{<\infty}))) \\
\Big\downarrow {\scriptstyle \mathrm{B}} & & \Big\downarrow {\scriptstyle \mathrm{B}} \\
\mathrm{Pro}(\mathrm{Grp}(\mathcal{X}_{<\infty})) & \longrightarrow & \mathrm{Grp}(\mathrm{Pro}(\mathcal{X}_{<\infty})),
\end{array}
$$

where the horizontal functors forget that a group is levelwise. The square commutes by naturality of the classifying sheaf functor B. The triangle commutes already before applying $\mathrm{Grp}(-)$: Indeed, all the involved functors preserve cofiltered limits, hence we may check this on constant objects, where it is clear. Let $G \in \mathrm{Grp}(\mathrm{Pro}(\mathcal{X}_{<\infty}))$ be pro-connected. Our goal is to show that G is in the essential image of the bottom horizontal functor. Chasing the commutative diagram, it suffices to show that the

right-hand composite sends G to itself. By Proposition 2.16, we see that $\mathrm{B}(\mathrm{Grp}(\Omega)(G)) \simeq \mathrm{Grp}(\mathrm{B}\Omega)(G)$. The result follows since $\mathrm{B}\Omega X \simeq X$ for any pro-connected $X \in \mathrm{Pro}(\mathcal{X}_{<\infty})_*$; indeed this was shown in Proposition 2.11(1). $\qquad\square$

2.5. **Group actions.** Let $\mathcal{C}$ be an ∞-category with pullbacks and I-indexed colimits. Given $B \in \mathrm{Fun}(I, \mathcal{C})$, the functor $\mathrm{colim}_I \colon \mathrm{Fun}(I, \mathcal{C})_{/B} \to \mathcal{C}_{/\,\mathrm{colim}_I B}$ admits a right adjoint $F \colon T \mapsto B \times_{d(\mathrm{colim}_I B)} d(T)$. Here $d \colon \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$ is the diagonal functor. By construction, F lands in the subcategory $\mathrm{Fun}(I, \mathcal{C})^{\mathrm{cart}}_{/B} \subset \mathrm{Fun}(I, \mathcal{C})_{/B}$ of *cartesian diagrams*, the full subcategory spanned by those natural transformations $\eta \colon B' \to B$ that are *cartesian* in the sense of [Lur09, Definition 6.1.3.1].

Now we specialize to $\mathcal{C} = \mathrm{Pro}(\mathcal{X}_{<\infty})$ and $I = \Delta^{\mathrm{op}}$. Let $G \in \mathrm{Grp}(\mathrm{Pro}(\mathcal{X}_{<\infty}))$. Recall that then $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Pro}(\mathcal{X}_{<\infty}))^{\mathrm{cart}}_{/\mathrm{B}_\bullet G}$ can be seen as encoding objects with a G-action [NSS14, Definition 3.1]. We obtain an adjunction

$$|-| \colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Pro}(\mathcal{X}_{<\infty}))^{\mathrm{cart}}_{/\mathrm{B}_\bullet G} \rightleftarrows \mathrm{Pro}(\mathcal{X}_{<\infty})_{/\mathrm{B}G} \colon F.$$

Proposition 2.17. *If G is a levelwise group, then the above adjunction is an adjoint equivalence.*

Proof. Let $\alpha = (X \to \mathrm{B}G) \in \mathrm{Pro}(\mathcal{X}_{<\infty})_{/\mathrm{B}G}$. Then $\alpha = \lim_i c\alpha_i$ in $\mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{X}_{<\infty}))$, where $\alpha_i \colon X_i \to B_i \in \mathcal{X}_{<\infty}$ with B_i connected (where $\lim_i B_i \simeq \mathrm{B}G$). Since geometric realizations commute with cofiltered limits in $\mathrm{Pro}(\mathcal{X}_{<\infty})$ (Proposition 2.2) we find $|F\alpha| \simeq \tau_{<\infty} \lim_i c|F_i\alpha_i|$, where

$$|-| \colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{X})^{\mathrm{cart}}_{/\mathrm{B}_\bullet \Omega B_i} \rightleftarrows \mathcal{X}_{/B_i} \colon F_i$$

is the canonical adjoint equivalence [Lur09, Theorem 6.1.3.9 (3)]. It follows that the counit $|F\alpha| \to \alpha$ is an equivalence.

We now prove the same for the unit. We first treat a special case. By Lemma 2.19 below, the forgetful functor $U \colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Pro}(\mathcal{X}_{<\infty}))^{\mathrm{cart}}_{/\mathrm{B}_\bullet G} \to \mathrm{Pro}(\mathcal{X}_{<\infty})$ has a left adjoint, sending $X \in \mathrm{Pro}(\mathcal{X}_{<\infty})$ to $G \times X$. Write $s \colon * \to \mathrm{B}G$ for the canonical map. Comparing universal properties we see that $|G \times X| \simeq s_\sharp X$, and so the object underlying $F|G \times X|$ is $* \times_{\mathrm{B}G} X \simeq F\mathrm{B}G \times X$, which by Proposition 2.11 is just $G \times X$. It follows that the unit $G \times X \to F|G \times X| \simeq G \times X$ is homotopic to the identity map, and so in particular an equivalence. To treat the general case, first observe that U is conservative and preserves geometric realizations (since geometric realizations commute with finite products in $\mathrm{Pro}(\mathcal{X}_{<\infty})$), hence is monadic, so any object in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Pro}(\mathcal{X}_{<\infty}))^{\mathrm{cart}}_{/\mathrm{B}_\bullet G}$ can be written as a geometric realization of objects of the form $G \times X$, cf. the proof of [Lur17, Proposition 4.7.3.14]. Since both $|-|$ and F preserve geometric realizations (the former being a left adjoint, and the latter by Proposition 2.2), we conclude. $\qquad\square$

If G is acting on an object $F \in \mathrm{Pro}(\mathcal{X}_{<\infty})$, by abuse of notation we also write F for the associated cartesian diagram $F_\bullet \to \mathrm{B}_\bullet G$, and write $F /\!\!/ G := |F^\bullet|$ for the homotopy quotient of the group action. By construction, this comes equipped with maps $F \to F /\!\!/ G \to \mathrm{B}G$.

Example 2.18. If G is a levelwise group acting on $F \in \mathrm{Pro}(\mathcal{X}_{<\infty})$, then $F \to F /\!\!/ G \to \mathrm{B}G$ is a fiber sequence. (This is a reformulation of the fact that the counit of the above adjunction is an equivalence.)

We used the following fact, for which we could not find a reference.

Lemma 2.19. *Let $\mathcal{C}$ be an ∞-category with finite products and $G \in \mathrm{Grp}(\mathcal{C})$. The forgetful functor $U \colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})^{\mathrm{cart}}_{/\mathrm{B}_\bullet G} \to \mathcal{C}$ has a left adjoint F such that for $X \in \mathcal{C}$ the counit $X \to UFX$ is given by the canonical map $X \simeq X \times * \to X \times G$.*

Proof. Increasing the size of the universe if necessary, we may assume that $\mathcal{C}$ is small. The claim implies that $(FX)_n \simeq G^n \times X$, so it suffices to prove it in the larger ∞-category $\mathcal{P}(\mathcal{C})$. We may thus assume that $\mathcal{C}$ is an ∞-topos. Now $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})^{\mathrm{cart}}_{/\mathrm{B}_\bullet G} \simeq \mathcal{C}_{/\mathrm{B}G}$ [Lur09, Theorem 6.1.3.9 (3)]. Writing $s \colon * \to \mathrm{B}G$ for the canonical map, the functor U identifies with s^* and so the left adjoint F is given by $s_\sharp$. It follows that $UFX \simeq X \times_{\mathrm{B}G} *$. The result follows by considering the following diagram of pullback squares

$$
\begin{array}{ccccc}
UFX & \longrightarrow & G & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & * & \longrightarrow & \mathrm{B}G.
\end{array}
$$

$\qquad\square$

3. Setup for the rest of the article

Throughout this article, we will work with the data of an adjunction of ∞-categories

$$L_+\colon \mathcal{X} \rightleftarrows \mathcal{D} \colon R.$$

Here $\mathcal{X}$ is an ∞-topos, $\mathcal{D}$ is stable and presentably symmetric monoidal, and L_+ is symmetric monoidal. We write $L\colon \mathcal{X}_* \to \mathcal{D}$ for the induced functor. We assume given $\mathcal{D}_{\geq 0}^{\mathrm{sld}} \subset \mathcal{D}$, which is the non-negative part of a t-structure (thus $\mathcal{D}_{\geq 0}^{\mathrm{sld}}$ is closed under colimits and extensions in $\mathcal{D}$). We denote by $\mathcal{D}^{\mathrm{sld}} \subset \mathcal{D}$ the localizing subcategory generated by $\mathcal{D}_{\geq 0}^{\mathrm{sld}}$ and call it the subcategory of *solid objects*. Objects in $\mathcal{D}_{\geq 0}^{\mathrm{sld}}$ are called *solid connective*. We are concerned with the following *axioms*:

(C) The restricted functor $R\colon \mathcal{D}_{\geq 0}^{\mathrm{sld}} \to \mathcal{X}$ preserves sifted colimits.

(M) For $X \in \mathcal{X}_*$ and $E \in \mathcal{D}_{\geq 0}^{\mathrm{sld}}$ we have $LX \otimes E \in \mathcal{D}_{\geq 0}^{\mathrm{sld}}$.

(S) (1) We have $LR(\mathcal{D}_{\geq 0}^{\mathrm{sld}}) \subset \mathcal{D}_{\geq 0}^{\mathrm{sld}}$.

  (2) For $E \in \mathcal{D}_{\geq 1}^{\mathrm{sld}}$ the canonical map $\pi_1 LRE \to \pi_1 E$ is an isomorphism. (Here π_1 refers to the t-structure specified by $\mathcal{D}_{\geq 0}^{\mathrm{sld}}$.)

  (3) $R\colon \mathcal{D}_{\geq 0}^{\mathrm{sld}} \to \mathcal{X}$ is conservative.

For convenience, we denote by $\mathcal{X}_{*,\geq n}^{\mathrm{sld}} \subset \mathcal{X}_*$ the subcategory of objects which are n-connective and such that $LX \in \mathcal{D}_{\geq n}^{\mathrm{sld}}$.

3.1. **First consequences.** The following result is standard:

Lemma 3.1 (Stable splitting). *Let $X, Y \in \mathcal{X}_*$. Then there is a split cofiber sequence*

$$LX \oplus LY \to L(X \times Y) \to L(X \wedge Y),$$

natural in X and Y.

Proof. Apply L to the cofiber sequence $X \vee Y \to X \times Y \to X \wedge Y$, and use that $L(X \vee Y) \simeq LX \oplus LY$ to construct a splitting of the inclusion. $\square$

Lemma 3.2. *The functor $R\colon \mathcal{D} \to \mathcal{X}$ factors canonically over $\mathrm{Grp}(\mathcal{X})$. Moreover, if axiom (C) is satisfied, then for every $E \in \mathcal{D}_{\geq 0}^{\mathrm{sld}}$, there is an equivalence $\mathrm{B}RE \simeq R\Sigma E$.*

Proof. Since $\mathcal{D}$ is stable, we have $\mathrm{Grp}(\mathcal{D}) \simeq \mathcal{D}$, and since R preserves products, we get an induced functor $\mathcal{D} \simeq \mathrm{Grp}(\mathcal{D}) \to \mathrm{Grp}(\mathcal{X})$ that refines R. For the second statement, note that ΣE can be computed as the geometric realization of the simplicial object

$$0 \Longleftarrow E \overset{\longleftarrow}{\Longleftarrow} E \oplus E \underset{\Longleftarrow}{\overset{\Longleftarrow}{\Longleftarrow}} \cdots,$$

see e.g. [BH21b, Lemma 2.7]. But under the canonical equivalence $\mathrm{Grp}(\mathcal{D}) \simeq \mathcal{D}$, this is just the bar construction of the group structure on E. Now, since R commutes with sifted colimits of objects in $\mathcal{D}_{\geq 0}^{\mathrm{sld}}$ by assumption, and E (and thus also $E^{\oplus n}$) is in $\mathcal{D}_{\geq 0}^{\mathrm{sld}}$, the result follows. $\square$

Remark 3.3. We will frequently use the following statement: Suppose that $X_\bullet$ is a simplicial object in a stable ∞-category $\mathcal{E}$ with a t-structure. If there exists an n such that $X_0 \in \mathcal{E}_{\geq n+1}$ and $X_i \in \mathcal{E}_{\geq n}$ for all $i > 0$, then $|X_\bullet| \in \mathcal{E}_{\geq n+1}$. Indeed, this follows immediately from [Lur17, Proposition 1.2.4.5].

Lemma 3.4. *(1) Axiom (C) implies that $R\mathcal{D}_{\geq n}^{\mathrm{sld}} \subset \mathcal{X}_{*,\geq n}$.*

  (2) Axioms (C) and (S1) together imply that $R\mathcal{D}_{\geq n}^{\mathrm{sld}} \subset \mathcal{X}_{*,\geq n}^{\mathrm{sld}}$.

  (3) Axiom (M) implies that for $X \in \mathcal{X}_{*,\geq n}$ and $E \in \mathcal{D}_{\geq m}^{\mathrm{sld}}$ we have $LX \otimes E \in \mathcal{D}_{\geq n+m}^{\mathrm{sld}}$.

  (4) Axioms (C), (S1) and (S3) together imply that every object in $\mathcal{D}_{\geq n}^{\mathrm{sld}}$ can be written as a sifted colimit of objects of the form LX, for $X \in \mathcal{X}_{*,\geq n}^{\mathrm{sld}}$.

  (5) Axioms (C), (S) and (M) together imply that for $E \in \mathcal{D}_{\geq n}^{\mathrm{sld}}$ with $n \geq 1$ we have $\mathrm{fib}(LRE \to E) \in \mathcal{D}_{\geq n+1}^{\mathrm{sld}}$.

Proof. (1) We prove the claim by induction on n, the case $n = 0$ being vacuous. Now let $E \in \mathcal{D}_{\geq n}^{\mathrm{sld}}$ with $RE \in \mathcal{X}_{*,\geq n}$; it suffices to show that $R\Sigma E \in \mathcal{X}_{*,\geq n+1}$. But $R\Sigma E \simeq \mathrm{B}RE$ by Lemma 3.2. The result follows from the fact that the bar construction takes n-connective objects to $(n+1)$-connective objects; see [Lur17, Remarks 5.2.2.19 and 5.2.6.17].

(2) Arguing inductively again, suppose $E \in \mathcal{D}_{\geq n}^{\mathrm{sld}}$ with $LRE \in \mathcal{D}_{\geq n}^{\mathrm{sld}}$. Then $LR\Sigma E$ is the geometric realization of a simplicial object $LRS_\bullet$ with $LRS_i = LR(E^{\oplus i}) \in \mathcal{D}_{\geq n}^{\mathrm{sld}}$. Since $LRS_0 = 0 \in \mathcal{D}_{\geq n+1}^{\mathrm{sld}}$, it follows that $|LRS_\bullet| \in \mathcal{D}_{\geq n+1}^{\mathrm{sld}}$ as needed (Remark 3.3).

(3) This is clear if $X \simeq \Sigma^n X'$. It follows in general since $\mathcal{X}_{*,\geq n}$ is generated under (sifted) colimits by objects of this form, see for example [Mat25, Proposition 2.39].

(4) Note that by definition and (2) the adjunction $L \dashv R$ restricts to an adjunction $L \colon \mathcal{X}_{*,\geq n}^{\mathrm{sld}} \rightleftarrows \mathcal{D}_{\geq n}^{\mathrm{sld}} \colon R$. Hence, the result follows from the proof of [Lur17, Proposition 4.7.3.14] if we can show that this induced adjunction is monadic. For this, it suffices to show that $R \colon \mathcal{D}_{\geq n}^{\mathrm{sld}} \to \mathcal{X}_{*,\geq n}^{\mathrm{sld}}$ is conservative and preserves sifted colimits. Conservativity follows immediately from (S3). Since both $\mathcal{D}_{\geq n}^{\mathrm{sld}} \subset \mathcal{D}_{\geq 0}^{\mathrm{sld}}$ and $\mathcal{X}_{*,\geq n}^{\mathrm{sld}} \subset \mathcal{X}_*$ are closed under (sifted) colimits (for the first, note that this is true for any t-structure, and for the second use that both n-connective objects, and objects that under L map to $\mathcal{D}_{\geq n}^{\mathrm{sld}}$ are closed under colimits), this is an immediate consequence of (C).

(5) We first treat the case $n = 1$. The functor $\mathcal{D}_{\geq 1}^{\mathrm{sld}} \to \mathcal{D}_{\geq 1}^{\mathrm{sld}}$, $E \mapsto \mathrm{fib}(LRE \to E)$ (which is well-defined by (S2) and (2)) preserves sifted colimits by (C). Thus, to prove that it takes values in $\mathcal{D}_{\geq 2}^{\mathrm{sld}}$, by (4) we need to check this only for objects of the form $E = LX$, where $X \in \mathcal{X}_{*,\geq 1}^{\mathrm{sld}}$. In this case the map $LRE \to E$ is split and so induces an epimorphism on π_2, as needed. Now we prove the case of general n by induction. Thus assume given $E \in \mathcal{D}_{\geq n}^{\mathrm{sld}}$ with $\mathrm{fib}(LRE \to E) \in \mathcal{D}_{\geq n+1}^{\mathrm{sld}}$. Consider again the simplicial object $S_\bullet$ from (2). We note that

$$LR(E \oplus E) \simeq L(RE \times RE) \simeq LRE \oplus (LRE \otimes LRE) \oplus LRE,$$

(using Lemma 3.1) with $LRE \otimes LRE \in \mathcal{D}_{\geq n+1}^{\mathrm{sld}}$ (use (1), (3) and recall $n \geq 1$), and similarly for the higher terms. It follows that the fiber of the canonical map $|LRS_\bullet| \simeq LR\Sigma E \to \Sigma E \simeq |S_\bullet|$ is the geometric realization of a simplicial object with entries $\mathrm{fib}(LRS_\bullet \to S_\bullet)$, i.e., a sum of "cross terms" (like $LRE \otimes LRE$) and copies of $\mathrm{fib}(LRE \to E)$. Both of these lie in $\mathcal{D}_{\geq n+1}^{\mathrm{sld}}$, and the degree zero term vanishes, whence by Remark 3.3 the geometric realization itself lies in $\mathcal{D}_{\geq n+2}^{\mathrm{sld}}$. This proves the claim. $\square$

Proposition 3.5. *Axiom (M) implies the following. Let $F \to E \to B$ be a fiber sequence, B connected and $F \in \mathcal{X}_{*,\geq n}^{\mathrm{sld}}$. Then $\mathrm{fib}(LE \to LB) \in \mathcal{D}_{\geq n}^{\mathrm{sld}}$. In particular, if $F, B \in \mathcal{X}_{*,\geq n}^{\mathrm{sld}}$ then also $E \in \mathcal{X}_{*,\geq n}^{\mathrm{sld}}$.*

Proof. Writing $E \simeq F /\!/ \Omega B$ and $B \simeq * /\!/ \Omega B$, we see that $E \to B$ is a colimit of maps of the form $\mathrm{pr}_X \colon F \times X \to X$. It will be enough to show that $\mathrm{fib}(L(F \times X) \to LX) \in \mathcal{D}_{\geq n}^{\mathrm{sld}}$. But by Lemma 3.1, $L(F \times X) \simeq LF \oplus LX \oplus (LF \otimes LX)$, so the fiber is $LF \oplus (LF \otimes LX)$, which lies in $\mathcal{D}_{\geq n}^{\mathrm{sld}}$ by (M). $\square$

Example 3.6. As a particular case of the last part of Proposition 3.5, we see that if $F, B \in \mathcal{X}_{*,\geq n}^{\mathrm{sld}}$ then also $F \times B \in \mathcal{X}_{*,\geq n}^{\mathrm{sld}}$.

4. HOMOLOGICAL LOCALIZATION

The left adjoint functor $L_+ \colon \mathcal{X} \to \mathcal{D}$ induces a Bousfield localization $\hat{L} \colon \mathcal{X} \to \mathcal{X}$ such that for a morphism α in $\mathcal{X}$, $L_+\alpha$ is an equivalence if and only if $\hat{L}\alpha$ is. This exists by the general theory of Bousfield localizations in presentable ∞-categories, as outlined in [Lur09, §5.5.4]. We say that $X \in \mathcal{X}$ is *L-complete* if it is a local object for $\hat{L}$, and that a morphism $f \colon X \to Y$ in $\mathcal{X}$ is an *L-equivalence* if Lf (or equivalently $\hat{L}f$) is an equivalence.

Remark 4.1. Note that $\hat{L}$ preserves the terminal object and hence defines an endofunctor of $\mathcal{X}_*$. Moreover, $\hat{L}|_{\mathcal{X}_*}$ is the Bousfield localization functor corresponding to $L \colon \mathcal{X}_* \to \mathcal{D}$. (Indeed given a morphism α in $\mathcal{X}_*$, we have $L(\alpha) \oplus L(\mathrm{id}_{S^0}) \simeq L_+(\alpha)$, so that $L(\alpha)$ is an equivalence if and only if $L_+(\alpha)$ is.)

Remark 4.2. The functor $\hat{L}$ preserves finite products, since L_+ is symmetric monoidal.

Theorem 4.3 (Bousfield–Kan fiber lemma). *Let $L_+ \colon \mathcal{X} \to \mathcal{D}$ be a left adjoint, symmetric monoidal functor from an ∞-topos to a presentably symmetric monoidal, stable ∞-category. Assume that $\mathcal{D}$ has a left-separated t-structure (i.e., has no nonzero ∞-connective objects) compatible with the symmetric monoidal structure such that $L_+(\mathcal{X}) \subset \mathcal{D}_{\geq 0}$.*

Let $F \to E \to B \in \mathcal{X}_*$ be a fiber sequence. Assume that B and $\hat{L}B$ are simply connected. Then $\hat{L}F \to \hat{L}E \to \hat{L}B$ is a fiber sequence.

Proof. We shall first prove the result under the additional assumption that $\hat{L}\Omega B \simeq \Omega\hat{L}B$. Set $\hat{F} \coloneqq \mathrm{fib}(\hat{L}E \to \hat{L}B)$. This is L-complete; it thus suffices to show that the canonical map $F \to \hat{F}$ is an L-equivalence, or equivalently an L_+-equivalence (Remark 4.1), i.e., that $L_+F \to L_+\hat{F}$ is an equivalence. Recall that F has an action by ΩB such that $E \simeq F /\!/ \Omega B$ [NSS14, Proposition 3.8], and similarly $\hat{L}E \simeq \hat{F} /\!/ \Omega\hat{L}B$. Moreover, since $F \to \hat{F}$ arises from a map of fiber sequences, this map intertwines the actions along $\Omega B \to \Omega\hat{L}B$. Since the latter map is in fact an L-equivalence (by our additional

assumption) and L_+ is a symmetric monoidal functor, we obtain an augmented $\mathcal{E}_1$-algebra $G = L_+\Omega B$ acting on $L_+F \to L_+\hat{F}$, with quotients given by L_+E and $L_+\hat{L}E \simeq L_+E$. In particular, G acts on $\operatorname{cof}(L_+F \to L_+\hat{F})$ with contractible quotient. Writing the quotient of the group action as geometric realization, we find that

$$0 = \left| \operatorname{cof}(L_+F \to L_+\hat{F}) \Leftarrow \operatorname{cof}(L_+F \to L_+\hat{F}) \otimes G \Lleftarrow \operatorname{cof}(L_+F \to L_+\hat{F}) \otimes G \otimes G \Llleftarrow \cdots \right|.$$

We will inductively show that $\operatorname{cof}(L_+F \to L_+\hat{F})$ is n-connective for every n; the case $n = 0$ holds trivially. So assume we know that $\operatorname{cof}(L_+F \to L_+\hat{F})$ is n-connective (and hence so are the other objects of the geometric realization). It now follows from [Lur17, Proposition 1.2.4.5] that

$$0 = \pi_n(0) \simeq \operatorname{Coeq}\left(\pi_n \operatorname{cof}(L_+F \to L_+\hat{F}) \otimes \pi_0(G) \rightrightarrows \pi_n \operatorname{cof}(L_+F \to L_+\hat{F}) \right).$$

Since by assumption ΩB is connected, $\pi_0(G) = \pi_0(L_+\Omega B) \simeq \pi_0(\mathbb{1})$, which implies that the two maps in the coequalizer are equal (in fact, they are isomorphic to the maps obtained by an analogous diagram where we act with $\mathbb{1}$ using the unit map $\mathbb{1} \to G$, i.e., the trivial action). Thus, $0 = \pi_n \operatorname{cof}(L_+F \to L_+\hat{F})$, concluding the induction. We showed that $\operatorname{cof}(L_+F \to L_+\hat{F})$ is ∞-connective, hence zero by assumption, as desired.

It remains to prove that $\hat{L}\Omega B \simeq \Omega\hat{L}B$, or equivalently, since $\Omega\hat{L}B$ is L-complete, that the canonical map $\Omega B \to \Omega\hat{L}B$ is an L-equivalence. Since B and $\hat{L}B$ are connected, $\mathrm{B}\Omega B \simeq B$ and $\mathrm{B}\Omega\hat{L}B \simeq \hat{L}B$. Since $B \to \hat{L}B$ is an L-equivalence by construction, we see that $\Omega B \to \Omega\hat{L}B$ becomes an L-equivalence after applying the bar construction. Write $X := \Omega B$, and $Y := \Omega\hat{L}B$. Thus, since L commutes with colimits, we see that

$$0 = |L\operatorname{cof}(\mathrm{B}^\bullet X \to \mathrm{B}^\bullet Y)| = \left| 0 \Leftarrow L\operatorname{cof}(X \to Y) \Lleftarrow L\operatorname{cof}(X \times X \to Y \times Y) \Llleftarrow \cdots \right|.$$

By assumption both X and Y are connected, whence $L\operatorname{cof}(X \to Y) \in \mathcal{D}_{\geq 1}$. We will inductively show that $L\operatorname{cof}(X \to Y) \in \mathcal{D}_{\geq n}$, which implies the claim since $\mathcal{D}$ is left-separated. So assume that $L\operatorname{cof}(X \to Y) \in \mathcal{D}_{\geq n}$ for some $n \geq 1$. For $k \geq 2$, the k-th term of the above geometric realization is also in $\mathcal{D}_{\geq n}$: Indeed, the k-th term is given by $L\operatorname{cof}(X^{\times k} \to Y^{\times k})$, which by stable splitting, Lemma 3.1, is the direct sum of the following objects:

- $L\operatorname{cof}(X \to Y)$,
- $L\operatorname{cof}(X^{\times k-1} \to Y^{\times k-1})$, and
- $L\operatorname{cof}(X \wedge X^{\times k-1} \to Y \wedge Y^{\times k-1})$.

The first object is in $\mathcal{D}_{\geq n}$ by assumption, and the second object by induction on k (or the assumption, if $k = 2$). The third object is even in $\mathcal{D}_{\geq n+1}$: applying the octahedral axiom to $X \wedge X^{\times k-1} \to Y \wedge X^{\times k-1} \to Y \wedge Y^{\times k-1}$ we obtain a cofiber sequence

$$L\operatorname{cof}(X \to Y) \otimes LX^{\times k-1} \to L\operatorname{cof}(X \wedge X^{\times k-1} \to Y \wedge Y^{\times k-1}) \to LY \otimes L\operatorname{cof}(X^{\times k-1} \to Y^{\times k-1})$$

where the outer terms are both in $\mathcal{D}_{\geq n+1}$ (by what we already concluded above and using that $LX^{\times k-1}$ and LY are in $\mathcal{D}_{\geq 1}$), and thus the same is true for the middle term.

Our next goal is to identify the three maps $L\operatorname{cof}(X \times X \to Y \times Y) \to L\operatorname{cof}(X \to Y)$. Since L commutes with the cofiber, and the stable splitting is functorial, it will suffice to analyze the three maps $L(X \times X) \to LX$ (and for Y, which is analogous). By definition of the bar construction, these maps are given by $L(\operatorname{pr}_1)$, $L(\operatorname{pr}_2)$ and $L(m)$, where $m: X \times X \to X$ is the multiplication map on the loop space. We claim that under the stable splitting equivalence, the following diagrams commute:

$$
\begin{array}{ccc}
 & L(X \times X) & \\
{\scriptstyle L(\operatorname{pr}_1)} \swarrow & \downarrow {\scriptstyle \simeq} & \searrow {\scriptstyle L(\operatorname{pr}_2)} \\
LX \xLeftarrow[(\mathrm{id},0,0)]{} & LX \oplus LX \oplus L(X \wedge X) & \xrightarrow[(0,\mathrm{id},0)]{} LX
\end{array}
\qquad
\begin{array}{ccc}
 & L(X \times X) & \\
 & \downarrow {\scriptstyle \simeq} & \searrow {\scriptstyle L(m)} \\
 & LX \oplus LX \oplus L(X \wedge X) & \xrightarrow[(\mathrm{id},\mathrm{id},\widetilde{m})]{} LX.
\end{array}
$$

Here, $\widetilde{m}$ is defined as the third component of the induced map. For the left diagram, this is true by construction. For the right diagram, it is only necessary to identify $L(m)$ on the first two components. Since the first inclusion $LX \to L(X \times X)$ is induced by the map $X = X \vee * \to X \times X$, and since m is the multiplication of a group object where the unit is given by $* \to X$, it follows that the composition

$X \to X \times X \xrightarrow{m} X$ is homotopic to the identity, even before applying L. The same argument works for the second component.

In particular, under the stable splitting equivalence, we see that $L(\mathrm{pr}_1) - L(m) + L(\mathrm{pr}_2) \simeq -\widetilde{m} \circ \mathrm{pr}_3$.

Consider now the associated spectral sequence of the above simplicial object $C_\bullet$, as explained in [Lur17, §1.2.4]. It has signature

$$E_1^{p,q} = H_p(\mathrm{DK}^{\mathrm{un}}(\pi_q(C_\bullet))) \implies \pi_{p+q}|C_\bullet| = 0,$$

where $\mathrm{DK}^{\mathrm{un}}$ denotes the associated unnormalized chain complex of a simplicial object in an abelian category. By the identification of the maps in the simplicial object done above, we see that for every q the associated unnormalized chain complex looks as follows:

$$0 \xleftarrow{0} \pi_q(L \operatorname{cof}(X \to Y)) \xleftarrow{-\pi_q(\operatorname{cof}(\widetilde{m}_X \to \widetilde{m}_Y))\mathrm{pr}_3} \pi_q(L \operatorname{cof}(X \to Y))^{\oplus 2} \oplus \pi_q(L \operatorname{cof}(X \wedge X \to Y \wedge Y)) \leftarrow \cdots.$$

Note that we use here that the map $X \to Y$ is a map of loop spaces, so that we get a map $\widetilde{m}_X \to \widetilde{m}_Y$. In particular,

$$H_1(\mathrm{DK}^{\mathrm{un}}(\pi_q(C_\bullet))) \simeq \operatorname{coker}(\pi_q(\operatorname{cof}(\widetilde{m}_X \to \widetilde{m}_Y))\colon \pi_q(L \operatorname{cof}(X \wedge X \to Y \wedge Y) \to \pi_q(L \operatorname{cof}(X \to Y)))).$$

By the connectivity estimates done above, we see that $\pi_n L \operatorname{cof}(X \wedge X \to Y \wedge Y) = 0$, whence $E_1^{1,n+1} = H_1(\mathrm{DK}^{\mathrm{un}}(\pi_{n+1}(C_\bullet))) \simeq \pi_n(L \operatorname{cof}(X \to Y))$. By inspection, no differential goes in or out of this group, therefore it survives to the E_∞-page. Since we know that the spectral sequence converges to 0, we get that $\pi_n(L \operatorname{cof}(X \to Y)) = 0$. This concludes our induction. $\square$

Remark 4.4. The above proof can be simplified somewhat by using Bar-Cobar duality. Specifically:

(1) The functor $(-) /\!/ G = (-) \otimes_G \mathbb{1}$ upgrades to an equivalence between G-modules and BG-comodules, thus is conservative.

(2) The functor $(-) \mapsto B(-) \simeq \mathbb{1} \otimes_{(-)} \mathbb{1}$ upgrades to an equivalence between augmented connected $\mathcal{E}_1$-algebras and coaugmented simply connected $\mathcal{E}_1$-coalgebras, so is conservative as well.

We included direct arguments in our special case to keep the proof more self-contained.

Remark 4.5. The assumption that $\hat{L}B$ is simply connected often does not apply for ∞-topoi of positive homotopy dimension. This restricts the usefulness of the above fiber lemma. However, it can be utilized to prove more general fiber lemmas by reducing to the case of a presheaf ∞-topos; see [Mat24b, Proposition 3.19] and [Mat25, Proposition 2.8]. Moreover, for rationalization it is always true (at least for ∞-topoi with enough points) that $\hat{L}B$ is still simply connected, cf. [Mat24a, Proposition 3.12].

5. BOUSFIELD–KAN COMPLETION

The main goal of the section is to prove a generalized version of Bousfield and Kan's Principal Fibration Lemma in our more general context (see Section 3). This is achieved in Theorem 5.7 which states that, assuming the axioms (C), (M), and (S), the generalized Bousfield–Kan completion functor preserves certain fiber sequences.

We begin the section by recalling how to define the functor of pro-objects $T^\bullet \colon \mathcal{X}_* \to \mathrm{Pro}(\mathcal{X}_*)$ associated to the adjunction $L \dashv R$. Our next order of business is to show that $T^\bullet$ preserves finite products (Proposition 5.1). So far we have not used any axioms. Making heavy use of the axioms, we next show that $\tau_{<\infty}T^\bullet$ preserves geometric realizations of connected solid object (Corollary 5.6) and that each of the functors T^n preserves connected solid objects (Corollary 5.4). The main theorem follows fairly straightforwardly from this.

5.1. **Setup.** Out of the adjunction $L\colon \mathcal{X}_* \rightleftarrows \mathcal{D} \colon R$ we construct an augmented cosimplicial endofunctor $\mathrm{CB}_+^\bullet \colon \mathcal{X}_* \to \mathrm{Fun}(\Delta_+, \mathcal{X}_*)$, see Construction A.1. Informally, $\mathrm{CB}_+^\bullet$ looks like

$$\mathrm{id} \longrightarrow RL \rightrightarrows (RL)^2 \underset{\longrightarrow}{\rightrightarrows} (RL)^3 \cdots.$$

Taking partial totalizations, we obtain

$$T^n = \mathrm{Tot}^n \mathrm{CB}^\bullet = \lim_{\Delta_{\leq n}} \mathrm{CB}^\bullet \colon \mathcal{X}_* \to \mathcal{X}_*$$

assembling into a tower

$$\cdots \to T^2 \to T^1 \to T^0.$$

We also denote by $T^\bullet$ the associated pro-object, i.e., $T^\bullet = \lim_n cT^n = \lim_\Delta c\,\mathrm{CB}^\bullet \colon \mathcal{X}_* \to \mathrm{Pro}(\mathcal{X}_*)$, which comes equipped with an augmentation $c \to T^\bullet$.

5.2. **Products.**

Proposition 5.1. *The functor* $T^{\bullet}\colon \mathcal{X}_* \to \mathrm{Pro}(\mathcal{X}_*)$ *preserves finite products.*

Proof. As $T^{\bullet}$ preserves the terminal object, it suffices to show that $T^{\bullet}$ preserves binary products. Let $X, Y \in \mathcal{X}_*$. Consider the augmented cosimplicial object

$$W_+^{\bullet}\colon \Delta_+ \xrightarrow{((\mathrm{CB})_+^{\bullet})^2} \mathrm{Fun}(\mathcal{X}_*, \mathcal{X}_*)^2 \xrightarrow{(\mathrm{ev}_X, \mathrm{ev}_Y)} \mathcal{X}_*^2 \xrightarrow{-\times-} \mathcal{X}_*.$$

Out of this, we can construct a double augmented cosimplicial object as follows:

$$V_+^{\bullet,\bullet}\colon \Delta_+ \times \Delta_+ \xrightarrow{\mathrm{CB}_+^{\bullet} \times W_+^{\bullet}} \mathrm{Fun}(\mathcal{X}_*, \mathcal{X}_*) \times \mathcal{X}_* \xrightarrow{\mathrm{eval}} \mathcal{X}_*.$$

Informally, $V_+^{\bullet,\bullet}$ looks like the following picture (where we do not draw the codegeneracy maps):

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\Uparrow\Uparrow\Uparrow & & \Uparrow\Uparrow\Uparrow & & \Uparrow\Uparrow\Uparrow & & \\
(RL)^2(X \times Y) \to (RL)^2(RLX \times RLY) \rightrightarrows (RL)^2((RL)^2X \times (RL)^2Y) \underset{\Rrightarrow}{\rightrightarrows} \cdots \\
\Uparrow\Uparrow & & \Uparrow\Uparrow & & \Uparrow\Uparrow & & \\
RL(X \times Y) \longrightarrow RL(RLX \times RLY) \rightrightarrows RL((RL)^2X \times (RL)^2Y) \underset{\Rrightarrow}{\rightrightarrows} \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
X \times Y \longrightarrow RLX \times RLY \rightrightarrows (RL)^2X \times (RL)^2Y \rightrightarrows \cdots
\end{array}
$$

By definition we have

$$\lim_{\Delta} c V^{\bullet, -1} = \lim_{\Delta} c\, \mathrm{CB}^{\bullet}(X \times Y) = T^{\bullet}(X \times Y).$$

Similarly, we have the following equivalence:

$$
\begin{aligned}
\lim_{\Delta} c V^{-1, \bullet} &= \lim_{\Delta} \mathrm{eval}(c, W^{\bullet}) \\
&= \lim_{\Delta} c(\mathrm{CB}^{\bullet}(X) \times \mathrm{CB}^{\bullet}(Y)) \\
&\simeq \lim_{\Delta} c\, \mathrm{CB}^{\bullet}(X) \times \lim_{\Delta} c\, \mathrm{CB}^{\bullet}(Y) \\
&= T^{\bullet}(X) \times T^{\bullet}(Y),
\end{aligned}
$$

where we used that c commutes with finite limits and that Δ is a cosifted category. Hence, we get a span

$$T^{\bullet}(X \times Y) \simeq \lim_{\Delta} c V^{\bullet, -1} \to \lim_{\Delta \times \Delta} c V^{\bullet, \bullet} \leftarrow \lim_{\Delta} c V^{-1, \bullet} \simeq T^{\bullet}(X) \times T^{\bullet}(Y).$$

The proposition follows if we can show that the arrows are equivalences. For this, it suffices to show that for every $n \geq 0$ both $V^{n, \bullet}$ and $V^{\bullet, n}$ are split cosimplicial objects, cf. Remark A.3. Unwinding the definitions, we see that

$$V_+^{n, \bullet} \simeq (RL)^{n+1}((\mathrm{CB})_+^{\bullet}(X) \times (\mathrm{CB})_+^{\bullet}(Y)).$$

This in turn is equivalent to

$$(RL)^n R \left(L(\mathrm{CB})_+^{\bullet}(X) \oplus L(\mathrm{CB})_+^{\bullet}(Y) \oplus \left(L(\mathrm{CB})_+^{\bullet}(X) \otimes L(\mathrm{CB})_+^{\bullet}(Y) \right) \right),$$

using Lemma 3.1. But now both $L(\mathrm{CB})_+^{\bullet}(X)$ and $L(\mathrm{CB})_+^{\bullet}(Y)$ are split by Proposition A.4, and so are their sums and tensor products. Similarly, we see that

$$V_+^{\bullet, n} \simeq (\mathrm{CB})_+^{\bullet} \left((RL)^{n+1}(X) \times (RL)^{n+1}(Y) \right) \simeq (\mathrm{CB})_+^{\bullet} \left(R(L(RL)^n(X) \times L(RL)^n(Y)) \right),$$

since R commutes with products as it is a right adjoint. This is also split by Proposition A.4. $\qquad\square$

5.3. **Connectivity.**

From now on to the end of the section, we will assume the axioms (C), (M), and (S). We shall be utilizing the *codegeneracy cubes* and the *face cubes*, whose total fibers we denote respectively by

$$C^n\colon \mathcal{X}_* \to \mathcal{X}_* \quad \text{and} \quad S^n\colon \mathcal{D} \to \mathcal{D}.$$

Informally speaking, the codegeneracy cubes are made out of the codegeneracy maps in the cosimplicial object $\mathrm{CB}^{\bullet}$, whereas the face cubes are made out of the face maps of the simplicial object extracted from the adjunction $L \dashv R$. For a careful definition of both cubes see Definition A.7. We use two main facts about these cubes:

- We have

(Cube1)
$$C^n \simeq RS^n L,$$

i.e., the codegeneracy maps in the cobar construction for X are just obtained by applying R to the face maps of the bar construction for LX. See Proposition A.10 for a reference.

- We have the fiber sequence

(Cube2)
$$\Omega^n C^n \to T^n \to T^{n-1},$$

i.e., C^n relates to an inductive construction of $T^\bullet$. (This follows from [Lur17, Corollary 1.2.4.18] together with the pointwise formula for right Kan extensions. Note that the stability assumption in the statement of *loc. cit.* is not actually used in the proof, just pointedness.)

Lemma 5.2. *The functor* $S^n \colon \mathcal{D}^{\mathrm{sld}}_{\geq 1} \to \mathcal{D}^{\mathrm{sld}}$ *sends* i-*connected maps to* $(n+i)$-*connected maps.*

Proof. Let (A_n) be the desired statement about S^n, let (B_n) be the (a priori weaker) statement that S^n sends $\mathcal{D}^{\mathrm{sld}}_{\geq i}$ to $\mathcal{D}^{\mathrm{sld}}_{\geq i+n}$ (for $i \geq 1$). We shall prove that (B_n) implies (A_n) and (A_n) implies (B_{n+1}). Since $S^0 = \mathrm{id}$, both (A_0) and (B_0) are trivial; hence by induction we will prove what we want.

$(A_n) \Rightarrow (B_{n+1})$: Let $E \in \mathcal{D}^{\mathrm{sld}}_{\geq i}$. The map $LRE \to E$ has fiber in $\mathcal{D}^{\mathrm{sld}}_{\geq i+1}$, by Lemma 3.4(5). Hence, by (A_n), the fiber of $S^n(LRE) \to S^n(E)$ lies in $\mathcal{D}^{\mathrm{sld}}_{\geq i+1+n}$. But this fiber is $S^{n+1}(E)$ by Proposition A.11, proving the claim.

$(B_n) \Rightarrow (A_n)$: Let $F \to X \to Y$ be a fiber sequence in $\mathcal{D}$ with $X, Y \in \mathcal{D}^{\mathrm{sld}}_{\geq 1}$ and $F \in \mathcal{D}^{\mathrm{sld}}_{\geq i}$. We write $Y \simeq \mathrm{cof}(F \to X)$ as the geometric realization of the simplicial object $Y_\bullet$ with entries $Y_t = X \oplus F^{\oplus t}$. Since S^n preserves sifted colimits, we find that $\mathrm{fib}(S^n(X) \to S^n(Y))$ is the geometric realization of the simplicial object $F_\bullet$, with $F_t = \mathrm{fib}(S^n(X) \to S^n(X \oplus F^{\oplus t}))$. Note that $S^n(A \oplus B) \simeq S^n(A) \oplus S^n(B) \oplus S^n(A, B)$, defining a new functor $S^n(-, -)$. In terms of this we have $F_t \simeq \Omega S^n(F^{\oplus t}) \oplus \Omega S^n(X, F^{\oplus t})$. In particular $F_0 = 0$. By (B_n) we know that $S^n(F^{\oplus t}) \in \mathcal{D}^{\mathrm{sld}}_{\geq n+i}$. Using Remark 3.3, it will thus suffice to show that $S^n(X, F^{\oplus t}) \in \mathcal{D}^{\mathrm{sld}}_{\geq n+i}$. We shall prove more generally that for $Z \in \mathcal{D}^{\mathrm{sld}}_{\geq i}$ we have $S^n(X, Z) \in \mathcal{D}^{\mathrm{sld}}_{\geq n+i}$. When $i = 1$ this follows from (B_n) ($S^n(X, Z)$ being a summand of $S^n(X \oplus Z)$). Inductively, assume the claim proved for i and let $\Sigma Z \in \mathcal{D}^{\mathrm{sld}}_{\geq i+1}$. Then $S^n(X, \Sigma Z) \simeq |S^n(X, Z^{\oplus \bullet})|$ (as $S^n(X, -)$ preserves sifted colimits, being a summand of $S^n(X \oplus -)$). Since each term in the simplicial object lies in $\mathcal{D}^{\mathrm{sld}}_{\geq i}$ (by induction) and the zeroth term is zero, the claim follows (again by Remark 3.3). $\square$

Proposition 5.3. *Let* $F \to X \to Y \in \mathcal{X}_*$ *be a fiber sequence with* $X, Y \in \mathcal{X}^{\mathrm{sld}}_{*, \geq 1}$ *and* $F \in \mathcal{X}^{\mathrm{sld}}_{*, \geq i}$ *(with* $i \geq 1$). *Then*
$$\mathrm{fib}(T^n(X) \to T^n(Y)) \in \mathcal{X}^{\mathrm{sld}}_{*, \geq i}.$$

Proof. Recall the fiber sequence (Cube2). By induction using Proposition 3.5, we thus reduce to showing that the functor $\Omega^n C^n$ has the desired property. Now $\Omega^n C^n \simeq R\Omega^n S^n L$ by (Cube1), so by Lemma 5.2 it is enough to show that $\mathrm{fib}(LX \to LY) \in \mathcal{D}^{\mathrm{sld}}_{\geq i}$. This is Proposition 3.5 again. $\square$

Corollary 5.4. *The functor* T^n *preserves* $\mathcal{X}^{\mathrm{sld}}_{*, \geq i}$ *(for* $i \geq 1$).

Proof. It is clear that $T^n(*) = *$, so this is the special case of Proposition 5.3 applied to a map to the terminal object. $\square$

5.4. Geometric realizations.

Proposition 5.5. *The functor* T^n *preserves sifted colimits of connected solid objects.*

Proof. We prove the claim by induction on n, the case $n = 0$ being clear since $T^0 = RL$ preserves sifted colimits of connective solid objects by axiom (C). Let $X_\bullet \in \mathcal{X}^{\mathrm{sld}}_{*, \geq 1}$ be a sifted diagram. By induction, we know that $|T^n(X_\bullet)| \simeq T^n(|X_\bullet|)$. We have

$$|\Omega^{n+1} C^{n+1}(X_\bullet)| \overset{(1)}{\simeq} |R\Omega^{n+1} S^{n+1}(LX_\bullet)| \overset{(2)}{\simeq} R|\Omega^{n+1} S^{n+1}(LX_\bullet)|$$
$$\overset{(3)}{\simeq} R\Omega^{n+1} S^{n+1}(L|X_\bullet|) \overset{(1)}{\simeq} \Omega^{n+1} C^{n+1}(|X_\bullet|),$$

where (1) holds by (Cube1), (2) holds because $\Omega^{n+1} S^{n+1}$ preserves solid connected objects (Lemma 5.2) and R preserves sifted colimits of solid connected objects (C), and (3) holds because $\Omega^{n+1} S^{n+1}$ is a total fiber computed in a stable ∞-category and hence preserves all colimits that are preserved by $(LR)^k$, in particular sifted colimits of solid connected objects. Recall the fiber sequence (Cube2). Using that pullbacks preserve sifted colimits in $\mathcal{X}$ with connected base [Lur17, Lemma 5.5.6.17], we see that

$|\Omega^{n+1}C^{n+1}(X_\bullet)| \to |T^{n+1}(X_\bullet)| \to |T^n(X_\bullet)|$ is a fiber sequence. We now have a morphism of fiber sequence

$$
\begin{array}{ccccc}
|\Omega^{n+1}C^{n+1}(X_\bullet)| & \longrightarrow & |T^{n+1}(X_\bullet)| & \longrightarrow & |T^n(X_\bullet)| \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{n+1}C^{n+1}(|X_\bullet|) & \longrightarrow & T^{n+1}(|X_\bullet|) & \longrightarrow & T^n(|X_\bullet|).
\end{array}
$$

The left- and right-hand maps are equivalence by what we have already said, and the base is connected, so we can conclude by using e.g. [NSS14, Proposition 3.8]. $\qquad\square$

Corollary 5.6. *The functor $\tau_{<\infty}T^\bullet$ preserves geometric realizations of connected solid objects.*

Proof. Immediate from Propositions 2.2 and 5.5. $\qquad\square$

5.5. **Fiber sequences.** We now have everything to prove our version of the celebrated Principal Fibration Lemma of Bousfield and Kan [BK72].

Theorem 5.7 (Principal Fibration Lemma). *Assume axioms (C), (M) and (S).*

Let $F \to E \to B \in \mathcal{X}_*$ be a fiber sequence with B simply connected and $F, \Omega B \in \mathcal{X}_{*,\geq 1}^{\mathrm{sld}}$. Then $\tau_{<\infty}T^\bullet F \to \tau_{<\infty}T^\bullet E \to \tau_{<\infty}T^\bullet B$ is a fiber sequence.

Proof. We know that ΩB is a group acting on F with quotient $F/\!\!/\Omega B \simeq E$ [NSS14, Proposition 3.8]. By Proposition 5.1, $\tau_{<\infty}T^\bullet\Omega B$ is a group acting on $\tau_{<\infty}T^\bullet F$. Note that $\Omega B^n \times F^\epsilon \in \mathcal{X}_{*,\geq 1}^{\mathrm{sld}}$ for all $n, \epsilon \geq 0$ (Example 3.6). Thus, Corollary 5.6 implies that the quotient is

$$
\tau_{<\infty}T^\bullet F /\!\!/ \tau_{<\infty}T^\bullet\Omega B \simeq \tau_{<\infty}T^\bullet(F/\!\!/\Omega B) \simeq \tau_{<\infty}T^\bullet E.
$$

Since $T^\bullet\Omega B$ is pro-connected by Corollary 5.4, it is a levelwise group by Theorem 2.15, and so by Example 2.18 we have a fiber sequence

$$
\tau_{<\infty}T^\bullet F \to \tau_{<\infty}T^\bullet E \simeq \tau_{<\infty}T^\bullet F /\!\!/ \tau_{<\infty}T^\bullet\Omega B \to \mathrm{B}\tau_{<\infty}T^\bullet\Omega B.
$$

Using once more that $\tau_{<\infty}T^\bullet$ preserves geometric realizations and finite products (of connected solid objects) we learn that $\mathrm{B}\tau_{<\infty}T^\bullet\Omega B \simeq \tau_{<\infty}T^\bullet\mathrm{B}\Omega B$, which is the same as $\tau_{<\infty}T^\bullet B$, as needed. $\qquad\square$

6. Applications

In this final section we reap the fruits of the previous sections to prove generalized versions of Bousfield and Kan's [BK72] convergence of their completion functor. We will start with an adjunction

$$
\Sigma_+^\infty : \mathcal{C} \rightleftarrows \mathcal{E} : \Omega^\infty,
$$

where $\mathcal{C}$ is some ∞-category of generalized spaces, and $\mathcal{E}$ is an associated ∞-category of spectra. We will pick $A \in \mathrm{CAlg}(\mathcal{E})$ and write $\mathcal{C}$ as a localization of an ∞-topos $\mathcal{X}$. Putting $\mathcal{D} = \mathcal{M}\mathrm{od}_A(\mathcal{E})$, we obtain a composed adjunction

$$
\mathcal{X} \rightleftarrows \mathcal{C} \rightleftarrows \mathcal{E} \rightleftarrows \mathcal{D}
$$

to which we shall apply the results of the previous sections.

There are four classes of examples for this broad framework that we have in mind. In Section 6.1 we consider the case $\mathcal{X} = \mathcal{C} = \mathcal{S}\mathrm{pc}$ the usual ∞-category of spaces, and $\mathcal{E} = \mathrm{Sp}$ the usual ∞-category of spectra. We thus provide an account of the original results of Bousfield and Kan. We will assume that A is connective. The behavior of the completion is then strongly influenced by $\pi_0(A)$, which we shall assume to be either a subring of $\mathbb{Q}$ of the form $S^{-1}\mathbb{Z}$, or a quotient of $\mathbb{Z}$, whence $\mathbb{Z}/n$. Depending on this, the Bousfield–Kan completion coincides in favorable cases (i.e., for nilpotent spaces) with either the unstable S-localization or the unstable n-completion.

Next in Section 6.2 we generalize somewhat, by allowing $\mathcal{C} = \mathcal{X}$ to be an ∞-topos satisfying some mild assumptions. We still set $\mathcal{E} = \mathrm{Sp}(\mathcal{X})$ and assume that $\underline{\pi}_0(A)$ is a constant sheaf $S^{-1}\underline{\mathbb{Z}}$ or $\underline{\mathbb{Z}}/n$. Again we prove that the Bousfield–Kan completion (or more precisely, its connected cover) agrees (for nilpotent sheaves) with an unstable S-localization or n-completion functor.

Our next example, treated in Section 6.3, is $\mathcal{C} = \mathcal{S}\mathrm{pc}(\mathrm{B}G)$, i.e. the ∞-category of genuine G-equivariant spaces. We only consider finite groups G. We can set $\mathcal{X} = \mathcal{C}$, since $\mathcal{S}\mathrm{pc}(\mathrm{B}G)$ is a presheaf topos. However, we no longer set $\mathcal{E} = \mathrm{Sp}(\mathcal{S}\mathrm{pc}(\mathrm{B}G))$. Indeed, as the practice of genuine equivariant stable homotopy theory shows, a much more useful ∞-category of spectra is $\mathcal{SH}(\mathrm{B}G)$, obtained by inverting on $\mathcal{S}\mathrm{pc}(\mathrm{B}G)_*$ all the representation spheres. We again pick $A \in \mathcal{SH}(\mathrm{B}G)_{\geq 0}$. But now $\pi_0(\mathbb{1}_{\mathcal{SH}(\mathrm{B}G)}) = \underline{A}$ (the Burnside ring Mackey functor) is no longer the constant presheaf $\underline{\mathbb{Z}}$, and so more options arise for $\underline{\pi}_0 A$. We treat four cases, (1) $\underline{\pi}_0 A = S^{-1}\underline{A}$, (2) $\underline{\pi}_0 A = \underline{A}/n$, (3) $\underline{\pi}_0 A = S^{-1}\underline{\mathbb{Z}}$ and (4) $\underline{\pi}_0 A = \underline{\mathbb{Z}}/p$. Cases

(1) and (2) behave very similarly to the previous situations, and we call them *Burnside resolutions.* Cases (3) and (4) are more delicate. We call them $\underline{\mathbb{Z}}$-*resolutions.*

Our final example is the ∞-category of motivic spaces $\mathcal{C} = \mathcal{S}\mathrm{pc}(k)$, where k is a perfect field; see Section 6.4. As in the equivariant case, the correct ∞-category of spectra is not the naive stabilization, but rather the ∞-category of motivic spectra $\mathcal{E} = \mathcal{SH}(k)$. Moreover, in contrast to the previous three examples, $\mathcal{C}$ is not a topos, and so we must specify $\mathcal{X} = \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$. As in the sheaves case, $\mathcal{X}$ has positive (even infinite) homotopy dimension, so care has to be taken with completions and connectivity assumptions. As in the equivariant case, $\underline{\pi}_0(\mathbb{1}_k)_*$ is more complicated than just $\underline{\mathbb{Z}}$, and so more options arise for $\underline{\pi}_0(A)_*$. Here we treat the cases $\underline{\pi}_0(A)_* = S^{-1}\underline{\pi}_0(\mathbb{1}_k)_*$, $\underline{\pi}_0(A)_* = \underline{\pi}_0(\mathbb{1}_k)_*/n$ (which we call *Milnor–Witt resolutions*) and $\underline{\pi}_0(A)_* = S^{-1}\underline{\pi}_0(\mathbb{1}_k)_*/\eta$, $\underline{\pi}_0(A)_* = \underline{\pi}_0(\mathbb{1}_k)_*/(n, \eta)$ (which we call *Milnor resolutions*).

6.1. **Spaces.** We set $\mathcal{X} = \mathcal{S}\mathrm{pc}$. In this subsection we explain how our results specialize to those of Bousfield–Kan [BK72]. Our presentation is designed as a drop-in replacement for *loc. cit.*

Let $A \in \mathrm{CAlg}(\mathrm{Sp})_{\geq 0}$ such that $\pi_0(A)$ is either a subring of $\mathbb{Q}$ or a quotient of $\mathbb{Z}$, so that $\pi_0(A) \otimes_{\mathbb{Z}} \pi_0(A) \simeq \pi_0(A)$. Set $\mathcal{D} = \mathcal{M}\mathrm{od}_A(\mathrm{Sp})$, $\mathcal{D}^{\mathrm{sld}}_{\geq 0} = \mathcal{D}_{\geq 0}$ and $L_+ = A \otimes \Sigma^{\infty}_+$. Hence, we have $\mathcal{S}\mathrm{pc}^{\mathrm{sld}}_{*,\geq n} \supseteq \mathcal{S}\mathrm{pc}_{*,\geq n}$ for all $n \geq 0$. The axioms are readily verified. (In fact, this is a special case of the situation of Section 6.2, where we verify the axioms more fully in Lemma 6.5.) In particular, for $X \in \mathcal{D}_{\geq 1}$ we have

$$\pi_1(LRX) \simeq \pi_1(X) \otimes_{\mathbb{Z}} \pi_0(A) \simeq \pi_1(X) \otimes_{\pi_0(A)} \pi_0(A) \otimes_{\mathbb{Z}} \pi_0(A) \simeq \pi_1(X),$$

whence (S2) is satisfied. To emphasize the dependence on A, we denote the localization functor (in the sense of Section 4) by $\hat{L}_A$ and the completion functor (in the sense of Section 5) by $T^{\bullet}_A$.

Let us recall the stable analog. On $\mathrm{Sp}_{\geq 0}$ we can define a Bousfield localization L^{st}_A, where $L^{st}_A(\alpha)$ is an equivalence if and only if $\alpha \otimes A$ is one. (In fact, for $E \in \mathrm{Sp}_{\geq 0}$, $L^{st}_A E$ coincides with the totalization of corresponding monadic resolution, see e.g. [Man24, Theorem 1.0.4], but we do not need this.) Its effect can be described explicitly, see Proposition B.1:

(1) If $\pi_0 A = \mathbb{Z}/n$ then $L^{st}_A E \simeq E^{\wedge}_n = \lim_i E/n^i$.
(2) If $\pi_0 A = S^{-1}\mathbb{Z} \subset \mathbb{Q}$ then $L^{st}_A E \simeq S^{-1}E$ (so that $\pi_*(L^{st}_A E) \simeq \pi_*(E) \otimes \pi_0(A)$).

We can now state the main result of this section.

Theorem 6.1 (Bousfield–Kan [BK72]). *(1) Let $X \in \mathcal{S}\mathrm{pc}_*$ be connected and nilpotent (i.e. $\pi_1(X)$ is nilpotent, with nilpotent action on $\pi_n(X)$ for all $n \geq 2$). The canonical map $\hat{L}_A X \to \mathrm{Tot}\, T^{\bullet}_A X$ is an equivalence. In fact, if $\pi_0(A) \subset \mathbb{Q}$ then already the canonical map $\tau_{<\infty}\hat{L}_A X \to \tau_{<\infty}T^{\bullet}_A X$ is an equivalence.*

(2) For $E \in \mathrm{Sp}_{\geq 1}$ the canonical map $\hat{L}_A \Omega^{\infty} E \to \Omega^{\infty} L^{st}_A E$ is an equivalence.

(3) Let $F \to E \to B$ be a fiber sequence of pointed, connected, nilpotent spaces, with B simply connected. Then $\hat{L}_A F \to \hat{L}_A E \to \hat{L}_A B$ is a fiber sequence.

(4) Let $X_{\bullet}$ be a tower of pointed, connected, nilpotent spaces with $X = \lim_i X_i$. Assume that the connectivity of $X \to X_i$ tends to infinity with i. Then $\hat{L}_A X \simeq \lim_i \hat{L}_A X_i$.

Note that this result supplies us with a recipe to compute $\hat{L}_A X$ (or equivalently $\mathrm{Tot}\, T^{\bullet}_A$) for X pointed, connected and nilpotent. Indeed, X being nilpotent, we can find fiber sequences $X_{i+1} \to X_i \to B_i$ where B_i is a simply connected Eilenberg–Mac Lane space, the connectivity of the B_i tends to infinity with i, and $X \simeq \lim_i X_i$. By (2) we know $\hat{L}_A B_i$ (in particular we know its homotopy groups), whence by (3) we can inductively understand $\hat{L}_A X_i$. Finally, by (4) we have $\hat{L}_A X \simeq \lim_i \hat{L}_A X_i$. The following formulae can be inductively deduced from the stable situation:

Example 6.2. Let $\pi_0 A \subset \mathbb{Q}$ and X be simply connected. Then $\pi_* \hat{L}_A X \simeq \pi_* X \otimes \pi_0 A$. In particular if $\pi_0 A = \mathbb{Z}$ then $\hat{L}_A X \simeq X$.

Example 6.3. Let $\pi_0 A = \mathbb{Z}/p$, X simply connected and $\pi_n X$ finitely generated for all n. Then $\pi_* \hat{L}_A X \simeq \pi_* X \otimes \mathbb{Z}^{\wedge}_p$.

The proof runs slightly differently depending on if $\pi_0 A \subset \mathbb{Q}$ or not, with the former case being somewhat easier. We treat this first.

Proof of Theorem 6.1 when $\pi_0 A \subset \mathbb{Q}$. We first prove (2). It is clear from the definitions that $\Omega^{\infty} L^{st}_A E$ is L-local. It thus suffices to show that $\Omega^{\infty} E \to \Omega^{\infty} L^{st}_A E$ is an L-equivalence. We can write $L^{st}_A E$ as the filtered colimit of a diagram of copies of E, with maps given by multiplication by elements in $\mathbb{Z} \cap \pi_0(A)^{\times}$. It thus suffices to show: if $E \in \mathrm{Sp}_{\geq 1}$ and $n \in \mathbb{Z} \cap \pi_0(A)^{\times}$, then $L\Omega^{\infty}(E \xrightarrow{n} E)$ is an equivalence. Observe that $L\Omega^{\infty} E \simeq \lim_i L\Omega^{\infty}(E_{\leq i})$: Indeed, the map $\Omega^{\infty} E \to \Omega^{\infty}(E_{\leq i})$ is $(i+1)$-connective, and L

preserves connectivity of maps (Proposition 3.5). In particular, the map $L\Omega^\infty E \simeq \lim_i L\Omega^\infty(E_{\leq i})$ has ∞-connective fiber, and thus is an equivalence. Therefore, we are reduced to the case that E is bounded. Appeal to Theorem 4.3 shows that given a fiber sequence $E_1 \to E_2 \to E_3$ with $E_3 \in \mathrm{Sp}_{\geq 2}$, the result for E_1 and E_3 implies the result for E_2. This way, working through the principalized Postnikov tower, we reduce to the case where E is concentrated in a single degree, whence, a connected $H\mathbb{Z}$-module. This ∞-category is generated under sifted colimits by $E = \Sigma H\mathbb{Z}$ itself, and so we reduce to this single case. Now $\Omega^\infty E \simeq S^1$ and $\Omega^\infty(n\colon E \to E)$ is indeed the degree n map from S^1 to itself. Upon stabilization, this corresponds to the multiplication by n map from the semiadditive structure (as this might be checked on $\pi_1(\Sigma^\infty S^1) \simeq \pi_1(S^1)$ by the Hurewicz theorem), and so becomes inverted upon $\otimes A$, as needed.

Theorem 4.3 now implies the special case of (3) when B is a simply-connected Eilenberg–Mac Lane space. From this we deduce by induction that if X is pointed, connected, nilpotent and truncated, then $\pi_* \hat{L}_A X \simeq \pi_* X \otimes \pi_0 A$.

Now we show a special case of (4), namely, the one where there are fiber sequences $X_{i+1} \to X_i \to B_i$ with B_i simply connected Eilenberg–Mac Lane spaces, with connectivity tending to infinity. Setting $X' = \lim_i \hat{L}_A X_i$, the key point is to show that the connectivity of $X' \to \hat{L}_A X_i$ tends to infinity with i. For this it suffices to show that the connectivity of $\hat{L}_A X_{i+1} \to \hat{L}_A X_i$ tends to infinity with i. But we have the fiber sequences $\hat{L}_A X_{i+1} \to \hat{L}_A X_i \to \hat{L}_A B_i$ (Theorem 4.3), so the claim follows.

Now we prove (1) for arbitrary (pointed, connected, nilpotent) X. Present $X \simeq \lim_i X_i$ with fiber sequences $X_{i+1} \to X_i \to B_i$ where B_i is a simply-connected Eilenberg–Mac Lane space and the connectivity of the B_i tends to infinity. Since the connectivity of $X \to X_i$ tends to infinity with i, so does the connectivity of $T^\bullet X \to T^\bullet X_i$ by Proposition 5.3. Consequently, $\tau_{<\infty} T^\bullet X \simeq \lim_i \tau_{<\infty} T^\bullet X_i$. Similarly, $\tau_{<\infty} \hat{L}_A X \simeq \lim_i \hat{L}_A X_i$ (via the special case of (4) above), and so it will be enough to show that $\hat{L}_A X_i \to \tau_{<\infty} T^\bullet X_i$ is an equivalence (in $\mathrm{Pro}(\mathcal{S}\mathrm{pc}_{<\infty})$). Since the fiber sequences $X_{i+1} \to X_i \to B_i$ are preserved by $\tau_{<\infty} T^\bullet$ (Theorem 5.7) and by $\hat{L}_A$ (via the special case of (3) above), we are reduced to proving the claim for the B_i. But, $T^\bullet B_i \simeq T^\bullet \hat{L}_A B_i$. Now note that $\hat{L}_A B_i \simeq \Omega^\infty \Sigma^k S^{-1} \pi_k(B_i)$ for some k. But $S^{-1}\pi_k(B_i)$ is a $(S^{-1}\mathbb{Z} = \pi_0(A))$-module, and hence also admits the structure of an A-module. Therefore, $\hat{L}_A B_i$ is in the image of R, and thus the cosimplicial object is split by Proposition A.4.

The general case of (3) is now an immediate consequence of Theorem 5.7 ($\tau_{<\infty} T^\bullet$ preserves fiber sequences), and the general case of (4) follows from Proposition 5.3 ($T^\bullet$ preserves connectivity). $\qquad\square$

Proof of Theorem 6.1 when $\pi_0 A = \mathbb{Z}/n$. We again first prove (2). To begin with, let $E \in \mathrm{Sp}_{\geq 2}$, and consider the fiber sequence $F \to E \to L_A^{st} E$. Then F is connected and $0 = L_A^{st} F = F_n^\wedge$, so that multiplication by n is an equivalence on F. As before $\Omega^\infty L_A^{st} E$ is L-local, and so it suffices to prove that $\Omega^\infty E \to \Omega^\infty L_A^{st} E$ is an L-equivalence. Writing $L_A^{st} E$ as the cofiber of $F \to E$, rewriting the cofiber as a geometric realization, using semiadditivity of Sp and the fact that Ω^∞ preserves sifted colimits, we find that $\Omega^\infty L_A^{st} E \simeq \Omega^\infty(E) /\!/ \Omega^\infty(F)$. Hence,

$$L_+ \Omega^\infty L_A^{st} E \simeq L_+ \Omega^\infty(E) /\!/ L_+ \Omega^\infty(F).$$

We have seen in the previous proof that multiplication by n is invertible on $\Sigma^\infty \Omega^\infty F$, and thus $L\Omega^\infty F = 0$. It follows that $L_+ \Omega^\infty F \simeq A$ and so $L_+ \Omega^\infty L_A^{st} E \simeq L_+ \Omega^\infty(E)$, as needed. Now suppose that $E \in \mathrm{Sp}_{\geq 1}$. Applying Theorem 4.3 to the fiber sequence $\Omega^\infty E \to * \to \Omega^\infty \Sigma E$ (note that we already know that $\hat{L}\Omega^\infty \Sigma E \cong \Omega^\infty L_A^{st} E$ is simply connected) we deduce that

$$\hat{L}_A \Omega^\infty E \simeq \Omega \hat{L}_A \Omega^\infty \Sigma E \simeq \Omega \Omega^\infty L_A^{st} \Sigma E \simeq \Omega^\infty L_A^{st} E,$$

as required.

As in the previous proof, this implies the special case of (3) when B is an Eilenberg–Mac Lane space. From this we deduce by induction that if X is pointed, d-connected, nilpotent and truncated, then $\hat{L}_A X$ satisfies the same assumptions.

Using the exact same argument as in the previous proof yields the special case of (4).

Next we prove (1) in the special case where X is a simply connected Eilenberg–Mac Lane space. Denote the functor $\mathrm{Tot}\, T^\bullet$ by F. Let B be an abelian group. For $i \geq 0$ we have $\Omega F K(B, i+3) \simeq F K(B, i+2)$, since F preserves fiber sequences (Theorem 5.7). We thus obtain an Ω-spectrum $G := (F K(B, i+2))_{i \geq 0}$ as well as a map of Ω-spectra $f\colon \Sigma^2 HB \to G$ (inducing in each level the map $K(B, i+2) \to F K(B, i+2)$). We claim that the fiber of multiplication by n on f is an equivalence. Indeed, by Theorem 5.7 again, the functor F preserves the fiber sequences $F_i \to K(B, i+2) \xrightarrow{n} K(B, i+2)$ and $F_i \to K(B/n, i+1) \to K(B[n], i+3)$. But both $B[n]$ and B/n are A-modules, whence $F K(B/n, i+1) \simeq K(B/n, i+1)$ and $F K(B[n], i+3) \simeq K(B[n], i+3)$ by Proposition A.4, from which we deduce $FF_i \simeq F_i$, proving the claim. Consequently, the map $\Sigma^2 HB \to G$ is an n-equivalence, so an L_A^{st}-equivalence, whence by (2) we

deduce that $\hat{L}_A K(B, i+2) \simeq \hat{L}_A F K(B, i+2)$. Since $FK(B, i+2)$ is clearly L-local, we have proved $FK(B, i+2) \simeq \hat{L}_A K(B, i+2)$, for any $i \geq 0$.

Finally, we prove the general case of (1). Writing $X = \lim_i X_i$ with $X_{i+1} \to X_i \to B_i$ and B_i simply connected Eilenberg–Mac Lane spaces of connectivity tending to ∞, as in the previous proof we find that $FX \simeq \lim_i FX_i$, and there are fiber sequences $FX_{i+1} \to FX_i \to FB_i$. Since the same holds for $\hat{L}_A$ in place of F by the special cases of (3) and (4) we established, showing that $FX \simeq \hat{L}_A X$ reduces to showing that $FB_i \simeq \hat{L}_A B_i$. This we treat just before.

The general cases of (3) and (4) follow as in the previous proof. $\qquad\square$

6.2. **Sheaves.** In this subsection we show the analog of Theorem 6.1 for (nice enough) ∞-topoi. Let $\mathcal{X}$ be an ∞-topos which admits a locally finite-dimensional cover [Mat24a, Definition 5.1]. In particular, $\mathcal{X}$ is Postnikov-complete [Mat24a, Lemma 5.3] and has enough points (this is part of the definition).

Example 6.4. There are two main examples of such $\mathcal{X}$: Any ∞-topos locally of homotopy dimension $\leq N$ (e.g., a presheaf ∞-topos) admits a locally finite-dimensional cover [Mat24a, Example A.1], and the Zariski, Nisnevich and étale ∞-topoi on smooth X-schemes, where X is some nice enough base scheme, cf. [Mat24a, Proposition A.3] and [Mat25, Proposition 2.25]. The existence of a locally finite-dimensional cover is a technical assumption that guarantees that certain homological localization functors (such as unstable p-completion) commute with the limit along (weak) Postnikov towers.

Let $A \in \mathrm{CAlg}(\mathrm{Sp}(\mathcal{X}))_{\geq 0}$ be a connective sheaf of commutative ring spectra, such that $\underline{\pi}_0(A)$ is a constant sheaf of commutative rings with values either a subring of $\mathbb{Q}$ or a quotient of $\mathbb{Z}$. As in the case of spaces we get $\underline{\pi}_0(A) \otimes_{\underline{\mathbb{Z}}} \underline{\pi}_0(A) \simeq \underline{\pi}_0(A)$. Set $\mathcal{D} = \mathcal{M}\mathrm{od}_A(\mathrm{Sp}(\mathcal{X}))$, $\mathcal{D}_{\geq 0}^{\mathrm{sld}} = \mathcal{D}_{\geq 0}$ and $L_+ = A \otimes \Sigma_+^\infty$. Therefore, we also get $\mathcal{X}_{*,\geq n}^{\mathrm{sld}} \supseteq \mathcal{X}_{*,\geq n}$ for all $n \geq 0$. Since $\mathcal{X}$ is Postnikov complete, it follows that the t-structure defined by $\mathcal{D}_{\geq 0}$ is left complete.

Lemma 6.5. *The axioms (C), (M) and (S) hold for the adjunction $L \dashv R$.*

Proof. The functor $R \colon \mathcal{D}_{\geq 0}^{\mathrm{sld}} \to \mathcal{X}$ is given by the composition $\mathcal{M}\mathrm{od}_A(\mathrm{Sp}(\mathcal{X})_{\geq 0}) \to \mathrm{Sp}(\mathcal{X})_{\geq 0} \to \mathcal{X}$; for the axioms (C) and (S3) it therefore suffices to show that both functors preserve sifted colimits and are conservative. For the first this is [Lur17, Proposition 4.8.5.8 (4) and Corollary 4.2.3.7 (2)]. For the second, since we have enough points, both statements can be checked on stalks, where the statements reduce to [Lur17, Corollary 5.2.6.27].

Axiom (M) follows since already $LX \in \mathcal{D}_{\geq 0}^{\mathrm{sld}}$ for all $X \in \mathcal{X}$, and $\mathcal{D}_{\geq 0}^{\mathrm{sld}}$ is stable under tensor products. The same argument shows axiom (S1).

For axiom (S2), let $E \in \mathcal{D}_{\geq 1}^{\mathrm{sld}}$. By the Hurewicz theorem we have $\underline{\pi}_1(\Sigma^\infty RE) \simeq \underline{\pi}_1(E)$ and thus

$$\underline{\pi}_1(LRE) \simeq \underline{\pi}_1(E) \otimes_{\underline{\mathbb{Z}}} \underline{\pi}_0(A) \simeq \underline{\pi}_1(E) \otimes_{\underline{\pi}_0(A)} \underline{\pi}_0(A) \otimes_{\underline{\mathbb{Z}}} \underline{\pi}_0(A) \simeq \underline{\pi}_1(E). \qquad\square$$

We can again identify the localization functors:

Lemma 6.6. *Let $E \in \mathrm{Sp}(\mathcal{X})_{\geq 0}$ and $X \in \mathcal{X}_*$ be nilpotent. If $\underline{\pi}_0(A) = S^{-1}\underline{\mathbb{Z}} \subset \underline{\mathbb{Q}}$, then*

(1) $L_A^{st} E \simeq S^{-1} E$, and

(2) $\hat{L}_A X \simeq L_S X$, where L_S is the unstable S-localization functor, cf. [Mat24a, §3].

Analogously, if $\underline{\pi}_0(A) = \underline{\mathbb{Z}}/n$, then

(1) $L_A^{st} E \simeq E_n^\wedge = \lim_k E/n^k = \prod_{p|n} E_p^\wedge$, and

(2) $\hat{L}_A X \simeq L_n X \simeq \prod_{p|n} L_p X$, the product over the unstable p-completion functors, cf. [Mat24b, §3].

Proof. For the stable identifications see Proposition B.1. We thus identify the unstable localization functors.

Suppose that $\underline{\pi}_0(A) = S^{-1}\underline{\mathbb{Z}}$. We first construct a natural transformation $L_S \to \hat{L}_A$, and then show that it is an equivalence on nilpotent sheaves. By definition, L_S is the Bousfield localization at morphisms $f_{U,n,k} \colon U \times (S^n \wedge S^1) \xrightarrow{\mathrm{id} \times \mathrm{id} \wedge k} U \times (S^n \wedge S^1)$ where $n \geq 0$, $U \in \mathcal{X}_*$, $k \in S$ and $k \colon S^1 \to S^1$ is multiplication by k. Hence, by the universal property, it suffices to show that $\hat{L}_A$ inverts those morphisms, i.e., that L inverts those morphisms. Using the stable splitting, Lemma 3.1, we reduce to showing that $A \otimes \Sigma^\infty(S^n \wedge S^1 \xrightarrow{\mathrm{id} \wedge k} S^n \wedge S^1)$ is an equivalence. This map is equivalent to $(A \xrightarrow{k} A) \otimes \Sigma^{n+1}\mathbb{1}$, which is an equivalence since $k \in S$. Thus, we get a natural transformation $\phi \colon L_S \to \hat{L}_A$, such that ϕ_X is a L-equivalence for all $X \in \mathcal{X}_*$. Let $X \in \mathcal{X}_*$ be nilpotent. In order to see that ϕ_X is an equivalence, it thus suffices to show that $L_S X$ is L-local. Since X is nilpotent, working through the layers of a principalized Postnikov tower using [Mat24b, Lemma A.15] as well as [Mat24a, Lemma 3.13, Lemma 3.18 and Proposition 6.9], we see that $L_S X$ lies in the closure under limits of $\Omega^\infty S^{-1} \mathrm{Sp}(\mathcal{X})_{\geq 2}$. We

conclude since the class of L-local objects is closed under limits, Ω^∞ converts L_A^{st}-local objects into L-local objects, and $\Omega^\infty S^{-1} \operatorname{Sp}(\mathcal{X})_{\geq 2} = \Omega^\infty L_A^{st} \operatorname{Sp}(\mathcal{X})_{\geq 2}$ (by (1)).

Suppose now that $\underline{\pi}_0(A) = \underline{\mathbb{Z}}/n$. Let $f: X \to Y \in \mathcal{X}$. By the stable identification, we know that f is an L-equivalence if and only if f is an n-equivalence in the sense of Definition C.2. Thus, we see that $\hat{L}_A \simeq L_n$, and the claim thus follows from Theorem C.6. $\qquad\square$

We need the following fact about highly connected towers:

Lemma 6.7. *Let $X \in \mathcal{X}_*$ be a sheaf, and let $(X_i)_i$ be a highly connected tower in $\mathcal{X}_*$ (in the sense of* [Mat24a, Definition 6.1], *i.e., the connectivity of $X_{i+1} \to X_i$ tends to ∞) under X. Then $X \to \lim_i X_i$ is an equivalence if and only if the connectivity of $X \to X_i$ tends to ∞.*

Proof. Assume first that $X \simeq \lim_i X_i$. We want to argue as in [Mat25, Lemma 2.4] that the connectivity of $X \to X_i$ tends to ∞. Unfortunately, the reference requires $\mathcal{X}$ to be locally of homotopy dimension $\leq N$ for some N. The same proof works in our case, but using [Mat24a, Corollary 6.6] instead of [Mat24a, Corollary 6.5].

Suppose on the other hand that the connectivity of $X \to X_i$ tends to ∞. Write $Y := \lim_i X_i$, so that there is a canonical map $X \to Y$. By the above, the connectivity of $Y \to X_i$ tends to ∞, hence, by two-out-of-three, the connectivity of $X \to Y$ tends to ∞. Since this is independent of i, we see that $X \to Y$ is ∞-connective, hence we conclude by hypercompleteness. $\qquad\square$

Theorem 6.8. *Let $\mathcal{X}$ be an ∞-topos which admits a locally finite-dimensional cover, and let $A \in \operatorname{CAlg}(\operatorname{Sp}(\mathcal{X})_{\geq 0})$ with $\underline{\pi}_0 A$ either a constant subsheaf of $\mathbb{Q}$ or a constant quotient sheaf of $\mathbb{Z}$.*

   (1) Let $X \in \mathcal{X}_*$ be nilpotent. The canonical map $\hat{L}_A X \to \tau_{\geq 1} \operatorname{Tot} T_A^\bullet X$ is an equivalence. In fact, if $\underline{\pi}_0(A) \subseteq \underline{\mathbb{Q}}$, then already the canonical map $\tau_{<\infty} \hat{L}_A X \to \tau_{<\infty} T^\bullet X$ is an equivalence.

   (2) For $E \in \operatorname{Sp}(\mathcal{X})_{\geq 1}$ the canonical map $\hat{L}_A \Omega^\infty E \to \tau_{\geq 1} \Omega^\infty L_A^{st} E$ is an equivalence. In fact, if $\underline{\pi}_0(A) \subseteq \underline{\mathbb{Q}}$, then already the canonical map $\hat{L}_A \Omega^\infty E \to \Omega^\infty L_A^{st} E$ is an equivalence.

   (3) Let $F \to E \to B$ be a fiber sequence of pointed nilpotent sheaves. Then the canonical map $\hat{L}_A F \to \tau_{\geq 1} \operatorname{fib}(\hat{L}_A E \to \hat{L}_A B)$ is an equivalence. In fact, if $\underline{\pi}_0(A) \subseteq \underline{\mathbb{Q}}$, then already $\hat{L}_A F \to \operatorname{fib}(\hat{L}_A E \to \hat{L}_A B)$ is an equivalence.

   (4) Let $X_\bullet$ be a tower of pointed, nilpotent sheaves with $X = \lim_i X_i$. Assume that the tower is locally highly connected subordinate to some cover $\mathcal{U}$ (in the sense of [Mat24a, Definition 6.1]*). Then the canonical map $\hat{L}_A X \to \lim_i \hat{L}_A X_i$ is an equivalence.*

Proof of Theorem 6.8 if $\underline{\pi}_0(A) = S^{-1}\mathbb{Z}$. Under the identifications of L_A^{st} and $\hat{L}_A$ from Lemma 6.6, (2), (3) and (4) follow from [Mat24a, Lemma 3.18, Lemma 3.13 and Proposition 6.9], respectively.

Now we prove (1), so let $X \in \mathcal{X}_*$ be nilpotent. Write $F := T_A^\bullet$. We can present $X \simeq \lim_i X_i$ with fiber sequences $X_{i+1} \to X_i \to B_i$ where B_i is an Eilenberg–Mac Lane space and the connectivity of the B_i tends to infinity, cf. [Mat24b, Lemma A.15]. By (3) and (4), the functor $\hat{L}_A$ preserves these fiber sequences and the limit appearing in the tower, and similarly F preserves these fiber sequences by Theorem 5.7. Noting that $(X_i)_i$ is a highly connected tower (under X) we deduce from Lemma 6.7 and Proposition 5.3 that the connectivity of $FX \to FX_i$ tends to infinity with i, even uniformly in the levels of the pro-object. By definition of $\tau_{<\infty}$, it follows that $\tau_{<\infty} FX \simeq \tau_{<\infty} \lim_i FX_i$. We have thus reduced to the case where $X = \Omega^\infty \Sigma^k M$ for some sheaf of abelian groups M and $k \geq 2$. Since both sides invert L-equivalences, we can replace X by $\hat{L}_A X$, and we have to see that $X \to \tau_{<\infty} FX$ is an equivalence. By (2) we know $X = \hat{L}_A \Omega^\infty \Sigma^k M \simeq \Omega^\infty \Sigma^k S^{-1} M$. The result follows from Proposition A.4, since $S^{-1} M$ is an $(S^{-1}\underline{\mathbb{Z}} = \underline{\pi}_0 A)$-module, whence an A-module. $\qquad\square$

Proof of Theorem 6.8 if $\underline{\pi}_0(A) = \mathbb{Z}/n$. We will use without mention the identifications of L_A^{st} and $\hat{L}_A$ from Lemma 6.6. Then (2) is Lemma C.16, (3) is Lemma C.17, and (4) is Lemma C.18.

We now establish (1). Since $\tau_{\geq 1}: \mathcal{X}_* \to \mathcal{X}_{*,\geq 1}$ is a right adjoint it preserves limits. Similarly, the functor $\hat{L}_A$ restricts to $\mathcal{X}_{*,\geq 1}$ (this follows from [Mat24b, Lemma 3.12]), and the restricted functor preserves appropriate fiber sequences (by Lemma C.17). Thus, as in the proof for $\underline{\pi}_0(A) = S^{-1}\underline{\pi}_0(\mathbb{1})$, we reduce to the case $X = \Omega^\infty \Sigma^k M$ an Eilenberg–Mac Lane sheaf, with $k \geq 2$. Write $F := \tau_{\geq 1} \operatorname{Tot} T_A^\bullet = \tau_{\geq 1} \operatorname{mat} \tau_{<\infty} T_A^\bullet$. Then

$$G(M) := (F \Omega^\infty \Sigma^k M, F \Omega^\infty \Sigma^{k+1} M, \dots) \in \operatorname{Sp}(\mathcal{X}_{*,\geq 1}) \simeq \operatorname{Sp}(\mathcal{X}),$$

since F preserves loops as above. The functor G also preserves fiber sequences of 2-connective spectra. There is a canonical map $\Sigma^k M \to G(M)$, which is an equivalence whenever M is a finite extension of A-modules, essentially by Proposition A.4. It follows that $\Sigma^k M \to G(M)$ is an n-equivalence, i.e.,

induces an equivalence on fibers of multiplication by n. Since $\Sigma^k M$ is connected, this implies that also $\Sigma^k M \to \tau_{\geq 1} G(M)$ is an n-equivalence (this follows easily from e.g. [Mat24b, Lemma 2.9 and Corollary 2.11]; note that in the proof there it is never used that p is a prime). Hence, using Lemma C.16, we see that

$$\Omega^\infty \Sigma^k M \to \Omega^\infty \tau_{\geq 1} G(M) \simeq F(\Omega^\infty \Sigma^k M)$$

is also an n-equivalence. Since the target is n-complete by construction, this concludes the proof. $\qquad\square$

6.3. **Equivariant spaces.** Let G be a finite group. We will now apply our machinery to the study of G-equivariant spaces, whose ∞-category we denote by $\mathcal{S}\mathrm{pc}(\mathrm{B}G)$. By Elmendorf's theorem [Elm83, Theorem 1] this is equivalent to an ∞-category of presheaves

$$\mathcal{S}\mathrm{pc}(\mathrm{B}G) \simeq \mathcal{P}_\Sigma(\mathrm{Fin}_G).$$

Here Fin_G denotes the 1-category of finite G-sets. In particular, $\mathcal{S}\mathrm{pc}(\mathrm{B}G)$ is a topos locally of homotopy dimension 0, so various results of the previous sections apply.

An additional complication which arises is that the appropriate notion of stabilization in this situation is not the naive one, i.e., $\mathcal{SH}^{S^1}(\mathrm{B}G) \coloneqq \mathrm{Sp}(\mathcal{S}\mathrm{pc}(\mathrm{B}G))$ only plays a minor role in the theory. Instead, the category of *genuine G-spectra* is obtained by inverting the regular representation sphere $S^G \in \mathcal{S}\mathrm{pc}(\mathrm{B}G)_*$, (see [LMSM86, §I.2] or [Cno24, Definition 4.1] for an ∞-categorical definition),

$$\mathcal{SH}(\mathrm{B}G) = \mathcal{S}\mathrm{pc}(\mathrm{B}G)_*[(S^G)^{-1}];$$

this fits into the usual adjunction

$$\Sigma^\infty \colon \mathcal{S}\mathrm{pc}(\mathrm{B}G)_* \rightleftarrows \mathcal{SH}(\mathrm{B}G) \colon \Omega^\infty.$$

A helpful alternative picture is the description as *spectral Mackey functors* [GM24, Nar16], i.e.,

$$\mathcal{SH}(\mathrm{B}G) \simeq \mathrm{Fun}^\times(\mathrm{Span}(\mathrm{Fin}_G)^{\mathrm{op}}, \mathrm{Sp}),$$

where $\mathrm{Span}(\mathrm{Fin}_G)$ denotes the $(2,1)$-category of spans in finite G-sets [BH21b, Appendix C]. The stabilization functor $\mathcal{S}\mathrm{pc}(\mathrm{B}G) \to \mathcal{SH}(\mathrm{B}G)$ factors over $\mathcal{SH}^{S^1}(\mathrm{B}G)$, cf. [Rob13, Corollary 4.24]. This in turn yields an adjunction $\sigma^\infty \colon \mathcal{SH}^{S^1}(BG) \rightleftarrows \mathcal{SH}(\mathrm{B}G) \colon \omega^\infty$. Alternatively the adjunction is obtained from the canonical functor $\mathrm{Fin}_G \to \mathrm{Span}(\mathrm{Fin}_G)$ by passing to spectral presheaves via left Kan extension and restriction. In particular, the left adjoint σ^∞ preserves compactly generating families by [Lur09, Proposition 5.5.8.10(6)] and construction, and hence ω^∞ is conservative and preserves all limits and colimits.

The above spectral Mackey functor description makes it clear, for example, that $\mathcal{SH}(\mathrm{B}G)$ has a t-structure with heart the abelian category of Mackey functors $\mathrm{Fun}^\times(\mathrm{Span}(\mathrm{Fin}_G)^{\mathrm{op}}, \mathrm{Ab})$. Given $X \in \mathcal{S}\mathrm{pc}(\mathrm{B}G)_*$ we write $\underline{\pi}_i(X) \in \mathrm{Fun}^\times(\mathrm{Fin}_G^{\mathrm{op}}, \mathrm{Set})$ for the zero-truncation of $\Omega^i X$, i.e.,

$$\underline{\pi}_i(X)(T) \simeq [T_+ \wedge S^i, X].$$

Similarly, given $E \in \mathcal{SH}(\mathrm{B}G)$ we denote by $\underline{\pi}_i(X) \in \mathrm{Fun}^\times(\mathrm{Span}(\mathrm{Fin}_G)^{\mathrm{op}}, \mathrm{Ab})$ the homotopy objects in the t-structure; one then has

$$\underline{\pi}_i E|_{\mathrm{Fin}_G} \simeq \underline{\pi}_0 \Omega^\infty \Sigma^{-i} E.$$

In order to apply our machinery in this situation, we need to construct a t-structure on $\mathcal{SH}(\mathrm{B}G)$ such that if E is 1-connective in the t-structure, then $\Sigma^\infty \Omega^\infty E \to E$ induces an isomorphism on the first homotopy object. Note that the above t-structure (with homotopy objects $\underline{\pi}_i E$) will not do (unless $G = 1$): by the tom Dieck splitting theorem [tD75, Satz 2] we see that for $E \in \mathcal{SH}(\mathrm{B}G)$ with $\underline{\pi}_i E = 0$ for $i \leq 0$ we have

$$\underline{\pi}_1(\Sigma^\infty \Omega^\infty E)(*) \simeq \bigoplus_{(H) \subset G} \underline{\pi}_1(E)(G/H)_{WH}.$$

Here, $WH \coloneqq N_G H/H$ denotes the Weyl group. This cannot agree with $\underline{\pi}_1(E)(*)$ unless $\underline{\pi}_1(E)(G/H) = 0$ for all proper subgroups H. Inspired by this, we make the following definition.

Definition 6.9. For $H \leq G$ we define $l(H) = \max\{r \mid H < H_1 < \cdots < H_r = G\}$, by convention $l(G) = 0$. Let $n \geq 0$. We define $X \in \mathcal{S}\mathrm{pc}(\mathrm{B}G)_*$ to be *(n,G)-connective* if for every $H \leq G$ and $i < n + l(H)$ we have $\underline{\pi}_i(X)(G/H) = *$. Similarly, for $n \in \mathbb{Z}$ we call $E \in \mathcal{SH}(\mathrm{B}G)$ *(n,G)-connective* if $\Omega^\infty \Sigma^i E$ is $(n+i, G)$-connective for all $i \geq -n$.

Remark 6.10. In other words, we require that $\mathrm{Map}_*(G/H_+, X)$ is $(n + l(H))$-connective, where $l(H)$ is the maximal length of a proper chain of subgroups of G starting at H. Note that $l(H)$ is bounded by the number of prime factors (with multiplicity) of $[G : H]$, and so in particular by $[G : H]$ itself.

Example 6.11. Since $\mathrm{Map}(S^0, S^G)$ is 1-connective (S^G decomposes as S^1 smash the reduced regular representation sphere) and $S^G|_{\mathcal{S}\mathrm{pc}(BH)} \simeq (S^H)^{\wedge[G:H]}$ we see that S^G is $(1, G)$-connective. More generally for $X \in \mathcal{S}\mathrm{pc}(BG)_*$, and $n \geq 1$ $(S^G)^{\wedge n} \wedge X$ is (n, G)-connective.

Lemma 6.12. *Let* $X \in \mathcal{S}\mathrm{pc}(BG)_*$ *and* $E \in \mathcal{SH}(BG)$ *be* (n, G)-*connective* $(n \geq 0$ *for* $X)$.

- (1) $\Sigma^\infty X$ *is* (n, G)-*connective.*
- (2) $\Omega^\infty E$ *is* (n, G)-*connective (provided* $n \geq 0$).
- (3) $\tau_{\geq i}X, \tau_{\leq i}X$ *are* (n, G)-*connective for* $i \geq 0$.
- (4) $\tau_{\geq i}E, \tau_{\leq i}E$ *are* (n, G)-*connective for* $i \in \mathbb{Z}$.
- (5) *Assuming* $n \geq 1$, *the fiber of* $\Sigma^\infty\Omega^\infty E \to E$ *is* $(n+1, G)$-*connective.*

Proof. (2) follows immediately from the definition, and (3) and (4) are clear since $\tau_{\geq i}, \tau_{\leq i}$ commute with fixed points (i.e. evaluating at G/H).

Let us prove (1). Let $X \in \mathcal{S}\mathrm{pc}(BG)$ be (n, G)-connective and $H \leq G$. For every $i < n + l(H)$ there is an equivalence $\underline{\pi}_i(\Sigma^\infty X)(G/H) \simeq \bigoplus_{(H') \leq H} \underline{\pi}_i(\Sigma^\infty(X(G/H')_{WH'}))$ by the tom Dieck splitting. Since by assumption $X(G/H')$ is $(n + l(H'))$-connective and $\Sigma^\infty : \mathcal{S}\mathrm{pc} \to \mathrm{Sp}$ preserves connectivity, (1) follows.

Now we prove (5). Write $F := \mathrm{fib}(\Sigma^\infty\Omega^\infty E \to E)$. As $\Sigma^\infty\Omega^\infty E$ is (n, G)-connective by (1) and (2), it follows that F is $(n-1, G)$-connective. It is left to show, that the morphisms $\underline{\pi}_{i+l(H)}(\Sigma^\infty\Omega^\infty E)(G/H) \to \underline{\pi}_{i+l(H)}(E)(G/H)$ are isomorphisms for all $H \leq G$ and $i = n$, and epimorphisms for $i = n + 1$. Since $n \geq 0$, we may as well check this after applying Ω^∞. But then $\Sigma^\infty\Omega^\infty E \to E$ splits, so the surjectivity (at $i = n + 1$) is clear. Finally, by the tom Dieck splitting, we have that

$$\underline{\pi}_{n+l(H)}(\Sigma^\infty\Omega^\infty E)(G/H) \simeq \underline{\pi}_{n+l(H)}(E)(G/H) \oplus \bigoplus_{(H') < H} \underline{\pi}_{n+l(H)}(E)(G/H')_{WH'}.$$

Now for $H' < H$ we have $l(H') > l(H)$, and so $\underline{\pi}_{n+l(H)}(E)(G/H')_{WH'} = 0$. This concludes the proof. $\square$

Now let $A \in \mathrm{CAlg}(\mathcal{SH}(BG)_{\geq 0})$ such that $\underline{\pi}_0(A)$ is idempotent. Furthermore, set $\mathcal{X} = \mathcal{S}\mathrm{pc}(BG)$, $\mathcal{D} = \mathcal{M}\mathrm{od}_A(\mathcal{SH}(BG))$, $L_+ = A \otimes \Sigma^\infty_+$ and

$$\mathcal{D}^{\mathrm{sld}}_{\geq 0} = \{E \in \mathcal{M}\mathrm{od}_A(\mathcal{SH}(BG)) \mid E \text{ is } (0, G)\text{-connective}\}.$$

Using Lemma 6.12(1), we have for all $n \geq$ that

$$\mathcal{S}\mathrm{pc}(BG)^{\mathrm{sld}}_{*, \geq n} \supseteq \{X \in \mathcal{S}\mathrm{pc}(BG)_* \mid X \text{ is } (n, G)\text{-connective}\}.$$

Let us observe the following about the interaction of the two t-structures.

Lemma 6.13. *Let* $E \in \mathcal{SH}(BG)_{\geq 0}$ *and* $F \in \mathcal{SH}(BG)^{\mathrm{sld}}_{\geq 0}$. *Then* $E \otimes F \in \mathcal{SH}(BG)^{\mathrm{sld}}_{\geq 0}$. *Moreover, for* $H \subset G$ *we have*

$$\pi_{l(H)}(E \otimes F)(G/H) \simeq \pi_0(\Phi^H E) \otimes \pi_{l(H)}(F(G/H)).$$

Proof. The first statement being stable under colimits in E, we may assume $E = \Sigma^\infty_+ G/H$. In this case $(\Sigma^\infty_+ G/H \otimes F)(G/H') \simeq F(G/H \times G/H')$, the orbit $\Sigma^\infty_+ G/H$ being self-dual. Now $G/H \times G/H'$ decomposes into a finite sum of G/H'', for H'' subconjugate to H'. Thus, $l(H'') \geq l(H')$ and so the connectivity of $F(G/H \times G/H')$ is at least as high as the connectivity of $F(G/H')$, as was to be shown.

For the final statement, we may assume $H = G$. Let $L_\Phi : \mathcal{SH}(BG) \to \mathcal{SH}(BG)$ denote the localization annihilating $\Sigma^\infty_+ G/H$ for all proper subgroups H. This is a smashing localization, $(-)^G$ induces an equivalence $L_\Phi \mathcal{SH}(BG) \simeq \mathrm{Sp}$, and $\Phi^G(-) = L_\Phi(-)^G$, essentially by definition, see [LMSM86, Corollary II.9.6] and [MNN17, Theorem 6.11]. Thus, we find

$$\pi_0((E \otimes F)(*)) \simeq \underline{\pi}_0(E \otimes F)(*) \simeq \underline{\pi}_0(E \otimes \tau_{\leq 0}F)(*) \simeq \underline{\pi}_0(L_\Phi(E) \otimes \tau_{\leq 0}F)(*) \simeq \pi_0(\Phi^G E) \otimes \pi_0(F(*)),$$

using that $\tau_{\leq 0}F \simeq L_\Phi\tau_{\leq 0}F$ (and L_Φ is smashing). $\square$

Lemma 6.14. *The axioms (C), (M) and (S) hold for the adjunction* $L \dashv R$.

Proof. Since $\mathcal{D}^{\mathrm{sld}}_{\geq 0}$ is accessible and closed under colimits and extensions, it is indeed the nonnegative part of a t-structure [Lur17, Proposition 1.4.4.11].

Consider the commutative diagram

$$\mathcal{SH}^{S^1}(BG)_{\geq 0} \xleftarrow{\omega^\infty} \mathcal{SH}(BG)_{\geq 0} \longleftarrow \mathcal{D}^{\mathrm{sld}}_{\geq 0}$$
$$\Omega^\infty \downarrow \qquad \swarrow \Omega^\infty$$
$$\mathcal{S}\mathrm{pc}(BG).$$

The functor R is given by the composition form the top right to the bottom left. For axioms (C) and (S3) it therefore suffices to show that all functors in the composition preserve sifted colimits and are conservative. The first one is monadic and preserves finite colimits, ω^∞ is conservative and preserves all colimits by the discussion above, and the claim for the vertical Ω^∞ is contained in the proof of Lemma 6.5.

Axiom (M) is immediate from Lemma 6.13.

Axiom (S1) is immediate from Lemma 6.12(1,2).

For (S2), let $E \in \mathcal{D}_{\geq 1}^{\mathrm{sld}}$. Using Lemma 6.12(5) and Lemma 6.13 we compute

$$\pi_{1+l(H)}(LRE)(G/H) \simeq \pi_{1+l(H)}(A \otimes E)(G/H) \simeq \pi_0(\Phi^H A) \otimes \pi_{1+l(H)}E(G/H).$$

Since $E \in \mathcal{SH}(\mathrm{B}G)_{\geq 1}^{\mathrm{sld}}$ is an A-module, $\pi_{1+l(H)}E(G/H)$ is a $(\Phi^H A)$-module. But $\underline{\pi}_0 A$ is idempotent, whence so is $\pi_0\Phi^H A$, and so $\pi_0(\Phi^H A) \otimes \pi_{1+l(H)}E(G/H) \simeq \pi_{1+l(H)}E(G/H)$, as needed. $\qquad\square$

We will study $\hat{L}_A$ and $\mathrm{Tot}\, T_A^\bullet$ for various choices of A in the next few subsections. They will be related to the usual unstable S-periodization and n-completion functors. This is based on the following.

Lemma 6.15. *Let $S \subset \mathbb{Z}$, and let $E \in \mathcal{SH}(\mathrm{B}G)$ be 1-connective. Denote by $L_S \colon \mathcal{S}\mathrm{pc}(\mathrm{B}G) \to \mathcal{S}\mathrm{pc}(\mathrm{B}G)$ the unstable S-localization functor. Then $L_S\Omega^\infty E \simeq \Omega^\infty S^{-1}E$.*

Similarly denote by $L_n \colon \mathcal{S}\mathrm{pc}(\mathrm{B}G) \to \mathcal{S}\mathrm{pc}(\mathrm{B}G)$ the unstable n-completion functor from Definition C.2. Then $L_n\Omega^\infty E \simeq \Omega^\infty E_n^\wedge$.

Proof. Since $\omega^\infty \colon \mathcal{SH}(\mathrm{B}G) \to \mathcal{SH}^{S^1}(\mathrm{B}G)$ preserves limits, colimits and 1-connective objects, we may replace $\mathcal{SH}(\mathrm{B}G)$ by $\mathcal{SH}^{S^1}(\mathrm{B}G)$ throughout. This being the stabilization of a presheaf ∞-topos, we conclude by [Mat24a, Lemma 3.18] and Lemma C.16. $\qquad\square$

Many of our arguments will rely on decomposing genuine G-spaces into infinite loop spaces of genuine G-spectra. The usual Postnikov tower is not sufficient for this. Indeed, if $X \in \mathcal{S}\mathrm{pc}(\mathrm{B}G)_*$, then $\underline{\pi}_i(X)$ is (for $i \geq 2$) a *coefficient system*, i.e., an object of $\mathrm{Fun}^\times(\mathrm{Fin}_G^{\mathrm{op}}, \mathrm{Ab})$. The corresponding Eilenberg–Mac Lane space is the infinite loop space of a genuine G-spectrum if and only if the coefficient system extends to a Mackey functor, which is not true in general. Luckily, we can actually build all coefficient systems out of Mackey functors, in an appropriate sense.

Lemma 6.16. *Let G be a finite group. Every coefficient system $C \in \mathrm{Fun}^\times(\mathrm{Fin}_G^{\mathrm{op}}, \mathrm{Ab})$ admits a resolution by Mackey functors*

$$0 \to C \to C^0 \to C^1 \to \cdots \to C^N \to 0,$$

i.e.:

- *Each C^i is the coefficient system underlying a Mackey functor.*
- *The above sequence of coefficient systems is exact.*

Moreover, the following hold:

- *The resolution is functorial in C.*
- *The number N is independent of C (in fact $N \leq |G|$).*
- *Let $H \leq G$ such that whenever $K \leq H$ we have $C(G/K) = 0$. Then also $C^i(G/H) = 0$.*

Proof. The adjunction $\sigma^\infty \dashv \omega^\infty$ can be restricted to an adjunction on the heart $F \colon \mathrm{Fun}^\times(\mathrm{Fin}_G^{\mathrm{op}}, \mathrm{Ab}) \rightleftarrows \mathrm{Fun}^\times(\mathrm{Span}(\mathrm{Fin}_G)^{\mathrm{op}}, \mathrm{Ab}) \colon U$, see [BBD82, Proposition 1.3.17(iii)]. In particular, this is the free-forgetful adjunction between coefficient systems and Mackey functors. For $C \in \mathrm{Fun}^\times(\mathrm{Fin}_G^{\mathrm{op}}, \mathrm{Ab})$ define $C^i := UF(\mathrm{coker}(C^{i-2} \to C^{i-1}))$ (set $C^{-1} := C, C^0 := 0$). This construction is clearly functorial in C. Exactness follows by construction from the fact that for any coefficient system D, the canonical map $D \to UFD$ is injective. In fact, by the tom Dieck splitting, we have

$$UFD(G/H) \simeq \bigoplus_{(H') \leq H} D(G/H')_{WH'}.$$

This also immediately proves the last bullet point. The only thing that remains to be done is to show that $C^N = 0$ for N large enough. For this, note that the tom Dieck splitting formula from above implies that if $D(G/H') = 0$ for all H' properly subconjugate to H, then $UFD(G/H) \simeq D(G/H)$ and so $(UFD/D)(G/H) = 0$. The construction of C^* thus implies that $C^i(G/H) = 0$ as soon as $i > l(H)$. $\qquad\square$

Now let $X \in \mathcal{S}\mathrm{pc}(\mathrm{B}G)_*$ be $(1, G)$-connective. Note that then $X(G/H)$ is simply connected for every proper subgroup H. From this it follows that X is nilpotent if and only if $X(*)$ is nilpotent. Applying a

functorial principalized Postnikov tower construction sectionwise, we can build a principalized Postnikov tower for X. Thus, we obtain fiber sequences

$$X_{i+1} \to X_i \to K(A_i, n_i + 1) \in \mathcal{S}\mathrm{pc}(\mathrm{B}G)_*.$$

Here each A_i is a coefficient system, $n_i \geq 1$, and there are compatible maps $X \to X_i$ of connectivity tending to infinity with i. Moreover, $\underline{\pi}_*(X_i)$ is a subquotient of $\underline{\pi}_*(X)$, and A_i is a subquotient of $\underline{\pi}_{n_i}(X)$. This implies that each X_i is $(1, G)$-connective and each $K(A_i, n_i + 1)$ is $(2, G)$-connective.

Construction 6.17. We refer to the data above as a $(1, G)$-connective principalized Postnikov tower for X.

Remark 6.18. Let A be a coefficient system and $n \geq 1$ such that $K(A, n)$ is $(2, G)$-connective. Rewriting the resolution of Lemma 6.16 as a system of short exact sequences, we obtain fiber sequences $K(A_i, n) \to K(M_i, n) \to K(A_{i+1}, n)$ with $A_1 = A$, M_i a Mackey functor and $A_i = 0$ for i sufficiently large. Moreover, each $K(A_i, n)$ and $K(M_i, n)$ are $(2, G)$-connective, by the last claim of that lemma.

Combining Construction 6.17 and Remark 6.18 yields our desired decomposition of nilpotent $(1, G)$-connective spaces into $(2, G)$-connective infinite loop spaces.

6.3.1. *Burnside Resolutions.* In this subsection, we assume that $\underline{\pi}_0(A)$ is either $S^{-1}\underline{\pi}_0(\mathbb{1})$ for $S \subset \mathbb{Z} \setminus 0$ or $\underline{\pi}_0(\mathbb{1})/n$ for $n \in \mathbb{N}$.

We can identify the associated localization functors.

Lemma 6.19. *Let $E \in \mathcal{S}\mathcal{H}(\mathrm{B}G)_{\geq 0}$ and $X \in \mathcal{S}\mathrm{pc}(\mathrm{B}G)_*$ nilpotent. If $\underline{\pi}_0(A) = S^{-1}\underline{\pi}_0(\mathbb{1})$, then*

  (1) $L_A^{st} E \simeq S^{-1}E$, and
  (2) $\hat{L}_A X \simeq L_S X$, where L_S is the unstable S-localization functor, cf. [Mat24a, §3].

  Analogously, if $\underline{\pi}_0(A) = \underline{\pi}_0(\mathbb{1})/n$, then
  (1) $L_A^{st} E \simeq E_n^\wedge = \lim_k E/n^k = \prod_{p|n} E_p^\wedge$, and
  (2) $\hat{L}_A X \simeq L_n X \simeq \prod_{p|n} L_p X$, the product over the unstable p-completion functors, cf. [Mat24b, §3].

Proof. The identification of L_A^{st} is Proposition B.1. The tom Dieck splitting theorem implies that $\sigma^\infty : \mathcal{S}\mathcal{H}^{S^1}(\mathrm{B}G) \to \mathcal{S}\mathcal{H}(\mathrm{B}G)$ is conservative. Suppose first that $\underline{\pi}_0(A) = S^{-1}\underline{\pi}_0(\mathbb{1})$. Then a map $f : Y \to Z$ of G-spaces is an A-equivalence if and only if $\Sigma_+^\infty f$ is an S-equivalence (by the stable identification), if and only if $\Sigma_{S^1}^\infty(f_+)$ is an S-equivalence (by conservativity). Thus, it follows from Lemma 6.6 that for $X \in \mathcal{S}\mathrm{pc}(\mathrm{B}G)_*$ nilpotent we have $\hat{L}_A X \simeq L_S X$. The case $\underline{\pi}_0(A) = \underline{\pi}_0(\mathbb{1})/n$ is treated similarly, additionally using Theorem C.6. □

Theorem 6.20. *(1) Let $X \in \mathcal{S}\mathrm{pc}(\mathrm{B}G)_*$ be nilpotent and $(1, G)$-connective. The canonical map $\hat{L}_A X \to \mathrm{Tot}\, T_A^\bullet X$ is an equivalence. In fact, if $\underline{\pi}_0(A) = S^{-1}\underline{\pi}_0(\mathbb{1})$, then already the canonical map $\tau_{<\infty} \hat{L}_A X \to \tau_{<\infty} T_A^\bullet X$ is an equivalence.*
  (2) For $E \in \mathcal{S}\mathcal{H}(\mathrm{B}G)_{\geq 1}$ the canonical map $\hat{L}_A \Omega^\infty E \to \Omega^\infty L_A^{st} E$ is an equivalence.
  (3) Let $F \to E \to B$ be a fiber sequence of pointed nilpotent G-spaces. Then the canonical map $\hat{L}_A F \to \mathrm{fib}\,(\hat{L}_A E \to \hat{L}_A B)$ is an equivalence.
  (4) Let $X_\bullet$ be a tower of pointed, nilpotent G-spaces with $X = \lim_i X_i$. Assume that the tower is highly connected. Then the canonical map $\hat{L}_A X \to \lim_i \hat{L}_A X_i$ is an equivalence.

Proof of Theorem 6.20 if $\underline{\pi}_0(A) = S^{-1}\underline{\pi}_0(\mathbb{1})$. Under the identifications of $\hat{L}_A$ and L_A^{st} from Lemma 6.19, (3) and (4) follow from [Mat24a, Lemma 3.13 and Proposition 6.9], respectively. Statement (2) then follows by combining Lemmas 6.15 and 6.19.

We now prove (1). Let X be nilpotent and $(1, G)$-connective. Working through a principalized Postnikov tower, as in the proof of Theorem 6.8, we can reduce to the case that $X = K(C, k)$ for a coefficient system C and $k \geq 2$. Note that as pointed out in Construction 6.17, the space $K(C, k)$ is $(2, G)$-connective. (A key point is that by Construction 6.17, we never leave the world of $(1, G)$-connective G-spaces, and so our axiomatics apply.) Using Lemma 6.16, Remark 6.18, and Theorem 5.7 (we can use the latter since in all the fiber sequences the base is $(2, G)$-connective), this reduces to the $(2, G)$-connective G-space $X = \Omega^\infty \Sigma^k M$, where M is a Mackey functor. Since both sides invert L-equivalences, we can replace X by $\hat{L}_A X$, and we have to see that $X \to \tau_{<\infty} T_A^\bullet X$ is an equivalence. By (2) we know $X = \hat{L}_A \Omega^\infty \Sigma^k M \simeq \Omega^\infty \Sigma^k S^{-1} M$. The result follows from Proposition A.4, since $S^{-1} M$ is an $(S^{-1}\underline{\pi}_0(\mathbb{1}) = \underline{\pi}_0(A))$-module, whence an A-module. □

Proof of Theorem 6.20 if $\underline{\pi}_0(A) = \underline{\pi}_0(\mathbb{1})/n$. We will use without mention the identifications of L_A^{st} and $\hat{L}_A$ from Lemma 6.19. Then (3) is Lemma C.17, and (4) is Lemma C.18. Moreover, (2) follows by combining Lemmas 6.15 and 6.19.

We now show (1). As in the proof for $\underline{\pi}_0(A) = S^{-1}\underline{\pi}_0(\mathbb{1})$, we reduce to the case that $X = \Omega^\infty \Sigma^k M$ is $(2, G)$-connective for a Mackey functor M. Write $F := \operatorname{Tot} T_A^\bullet$. Then

$$P(M) := (F\Omega^\infty \Sigma^k M, F\Omega^\infty \Sigma^{k+1} M, \dots) \in \mathcal{SH}^{S^1}(\mathrm{B}G),$$

since F preserves loops by Theorem 5.7. The functor P also preserves fiber sequences of 2-connective spectra. There is a canonical map $\omega^\infty \Sigma^k M \to P(M)$, which is an equivalence whenever M is a finite extension of A-modules, essentially by Proposition A.4. It follows that $\Sigma^k M \to P(M)$ is an n-equivalence, i.e., induces an equivalence on fibers of multiplication by n. Hence, using Lemma C.16, we see that

$$\Omega^\infty \Sigma^k M \to \Omega^\infty P(M) \simeq F(\Omega^\infty \Sigma^k M)$$

is also an n-equivalence. Since the target is n-complete by construction, this concludes the proof. $\qquad\square$

6.3.2. *$\mathbb{Z}$ Resolutions: Localization.* Let G be a finite group and I the augmentation ideal of the Burnside ring Mackey functor $\underline{A}$, i.e., the kernel of the surjective rank map $\underline{A} \to \underline{\mathbb{Z}}$. Throughout this subsection we assume that $\underline{\pi}_0(A) \simeq S^{-1}(\underline{\pi}_0(\mathbb{1})/I) \simeq S^{-1}\underline{\mathbb{Z}}$ for $S \subset \mathbb{Z} \setminus 0$ and $|G| \in S$. Consider the element $e_e^G = [G/e]/|G| \in A(G)[1/|G|]$. This is immediately checked to be an idempotent. For $E \in S^{-1}\mathcal{SH}(\mathrm{B}G)$ we put

$$(e_e^G)^{-1}E = \operatorname{colim} E \xrightarrow{e_e^G} E \xrightarrow{e_e^G} \cdots .$$

We define $(1 - e_e^G)^{-1}E$ similarly. Then the canonical map $E \to (e_e^G)^{-1}E \oplus (1 - e_e^G)^{-1}E$ is an equivalence, as is immediately verified on homotopy Mackey functors.

Remark 6.21. Note that $(1 - e_e^G)^{-1}E = 0$ if and only if $[G]\underline{\pi}_*E = 0$, i.e., $E(G/e) = 0$.

To identify the localization $\hat{L}_A$, it will be useful to consider the inclusion $i : \mathrm{B}G \to \mathrm{Fin}_G$ sending the unique object of the source to G. This induces an adjunction

$$i^* : \mathcal{S}\mathrm{pc}(\mathrm{B}G) \rightleftarrows \operatorname{Fun}(\mathrm{B}G, \mathcal{S}\mathrm{pc}) : i_*.$$

Here i^*X is just $X(G)$ viewed as a naive G-space, and i_* turns a naive G-space into a genuine one by declaring the genuine fixed points to be the homotopy fixed points. In particular, i_* is fully faithful. (The functor i^* has a further left adjoint $i_!$, which does not concern us here.) We also have

$$i^* : \mathcal{SH}(\mathrm{B}G) \rightleftarrows \operatorname{Fun}(\mathrm{B}G, \mathrm{Sp}) : i_*,$$

with similar properties, e.g. [NS18, Theorem II.2.7].

Lemma 6.22. *Let $E \in \mathcal{SH}(\mathrm{B}G)$ be bounded below and $X \in \mathcal{S}\mathrm{pc}(\mathrm{B}G)_*$ be nilpotent.*
   (1) We have $L_A^{st}E \simeq (e_e^G)^{-1}S^{-1}E$.
   (2) Alternatively, we have $L_A^{st}E \simeq i_*S^{-1}i^*E$.
   (3) We have $\hat{L}_A X \simeq i_*L_S i^*X$.

Proof. (1) The morphism $\underline{A} \to \underline{A}/I$ admits a section given by $1 \mapsto [G/e]$, implying that $S^{-1}\underline{A}[1/e_e^G] \xrightarrow{\simeq} S^{-1}\underline{A}/I \simeq \underline{\pi}_0(A)$.

The previous sentence shows that e_e^G maps to a unit in $\underline{\pi}_0(A)$ and so $L_A^{st}E \simeq L_A^{st}(e_e^G)^{-1}S^{-1}E$. It will thus be enough to show that $E' := (e_e^G)^{-1}S^{-1}E$ is A-local. This follows from a Postnikov tower argument, using that E' is bounded below (since E is) and each $\underline{\pi}_i(E')$ as a module over $(e_e^G)^{-1}S^{-1}\underline{A} \simeq \underline{\pi}_0(A)$.

(2) Since i_* is fully faithful, $i_*S^{-1}i^*$ is a localization functor, namely at the underlying S-equivalences. By Remark 6.21, so is $(e_e^G)^{-1}S^{-1}$. The claim thus follows from (1).

(3) By (2), a map f in $\mathcal{S}\mathrm{pc}(\mathrm{B}G)$ is an L-equivalence if and only if $S^{-1}\Sigma_+^\infty i^*f$ is an equivalence. In particular, $\hat{L}_A = \hat{L}_A i_* i^*$, the functor i_* being fully faithful (i.e., part of a localization). The subcategory of objects of the form $i_*(-)$ is equivalent to $\operatorname{Fun}(\mathrm{B}G, \mathcal{S}\mathrm{pc})$, and under this equivalence, $\hat{L}_A$ on this subcategory just coincides with localization at the homological S-equivalences. For nilpotent objects, this is given by L_S (Lemma 6.6). This concludes the proof. $\qquad\square$

Theorem 6.23.   *(1) Let $X \in \mathcal{S}\mathrm{pc}(\mathrm{B}G)_*$ be nilpotent and $(1, G)$-connective. The canonical map $\tau_{<\infty}\hat{L}_A X \to \tau_{<\infty} T_A^\bullet X$ is an equivalence.*
   (2) For $E \in \mathcal{SH}(\mathrm{B}G)_{\geq 1}$ the canonical map $\hat{L}_A \Omega^\infty E \to \Omega^\infty L_A^{st}E$ is an equivalence.
   (3) Let $F \to E \to B$ be a fiber sequence of pointed nilpotent $(1, G)$-connective G-spaces. Then $\hat{L}_A F \simeq \operatorname{fib}(\hat{L}_A E \to \hat{L}_A B)$.

(4) Let $X_\bullet$ be a tower of pointed nilpotent $(1, G)$-connective G-spaces with $X = \lim_i X_i$. Assume that the tower is highly connected. Then $\hat{L}_A X \simeq \lim_i \hat{L}_A X_i$.

Proof. (2), (3) and (4) are immediate from Lemma 6.22, the fact that i^* and i_* preserve all limits (since they are right adjoints), and the corresponding facts about L_S, cf. [Mat24a, Lemma 3.18, Lemma 3.13 and Proposition 6.9].

For statement (1), arguing in the usual way (see e.g. Theorem 6.20(1)), we reduce the case $X = K(M, i)$ where $i \geq 2$ and M is Mackey functor with $M \simeq i_* i^* S^{-1} M$. But then M is a module over $S^{-1}\underline{\mathbb{Z}}$, and so the claim follows from Proposition A.4. $\qquad\square$

6.3.3. $\underline{\mathbb{Z}}$ *Resolutions: Completion.* For this subsection let G be a finite p-group. We continue to write I for the augmentation ideal of the Burnside ring Mackey functor $\underline{A}$. Assume that $\underline{\pi}_0(A) \simeq \underline{A}/(I, p) \simeq \underline{\mathbb{F}}_p$.

Lemma 6.24. *Let $E \in \mathcal{SH}(\mathrm{B}G)_{\geq 0}$. Then $L_A^{st} E \simeq E_p^\wedge$.*

Moreover, $\hat{L}_A X \simeq L_p X$ for $X \in \mathcal{S}\mathrm{pc}(\mathrm{B}G)_*$ nilpotent.

Proof. To show the stable identification use that $L_A^{st} E \simeq E_{I,p}^\wedge$ by Proposition B.1. Since $E_{I,p}^\wedge$ is p-complete it suffices to show that $E_p^\wedge$ is $\underline{\mathbb{F}}_p$-complete. Let $(E_{\leq n})_n$ be a Postnikov tower with $E \simeq \lim_n E_{\leq n}$. It is enough to show that all $(E_{\leq n})_p^\wedge$ are $\underline{\mathbb{F}}_p$-complete as completion commutes with limits, e.g. [Mat24b, Lemma 2.5]. Using the fiber sequence $(E_{=n})_p^\wedge \to (E_{\leq n})_p^\wedge \to (E_{<n})_p^\wedge$ we reduce to the case of $\Sigma^k M$, where M is a Mackey functor. We have that $M_p^\wedge \simeq \lim_n M/\!\!/p^n$ and moreover a fiber sequence $M/\!\!/p^n \xrightarrow{p} M/\!\!/p^{n+1} \to M/\!\!/p$. Hence, it suffices to show that $M/\!\!/p$ is $\underline{\mathbb{F}}_p$-local. There is a fiber sequence $\Sigma(M[p]) \to M/\!\!/p \to M/p$. Therefore, we may reduce to the case where M is p-torsion and thus a $(\underline{\pi}_0(\mathbb{1})/p)$-module. It is even a $(\underline{\pi}_0(\mathbb{1})/I^n)$-module for some $n \geq 0$ by [Lai79, Proposition 1.12] (it is here where we need G to be a finite p-group). For $k \geq 0$ we obtain exact sequences $I^k M \to I^{k-1} M \to I^{k-1} M/I^k M$. Since each $I^{k-1} M/I^k M$ is an $\underline{A}/I$-module in addition to being an $\underline{A}/p$-module, it is in fact an $\underline{\mathbb{F}}_p$-module. Since $I^n M = 0$ it follows that M is $\underline{\mathbb{F}}_p$-local. This shows the claim in the stable case.

The unstable claim follows by the same argument as in Lemma 6.19. $\qquad\square$

Theorem 6.25. (1) *Let $X \in \mathcal{S}\mathrm{pc}(\mathrm{B}G)_*$ be nilpotent and $(1, G)$-connective. The canonical map $\hat{L}_A X \to \mathrm{Tot}\, T^\bullet X$ is an equivalence.*

(2) *For $E \in \mathcal{SH}(\mathrm{B}G)_{\geq 1}$ the canonical map $\hat{L}_A \Omega^\infty E \to \Omega^\infty L_A^{st} E$ is an equivalence.*

(3) *Let $F \to E \to B$ be a fiber sequence of pointed nilpotent G-spaces. Then the canonical map $\hat{L}_A F \to \mathrm{fib}\left(\hat{L}_A E \to \hat{L}_A B\right)$ is an equivalence.*

(4) *Let $X_\bullet$ be a tower of pointed, nilpotent G-spaces with $X = \lim_i X_i$. Assume that the tower is highly connected. Then the canonical map $\hat{L}_A X \to \lim_i \hat{L}_A X_i$ is an equivalence.*

Proof. (3) and (4) where already shown in Theorem 6.20. For (2) combine Lemmas 6.15 and 6.24.

For (1) we repeat the arguments from Theorem 6.20(1) in the case $\underline{\pi}_0 A = \underline{\pi}_0(\mathbb{1})/p$. We must eventually show that $\omega^\infty \Sigma^k M \to G(M)$ is a p-equivalence for Mackey functors M, and we know this if M is a module over $\underline{\mathbb{F}}_p$. As in that proof, one reduces to the case where M is p-torsion. As in the proof of Lemma 6.24, this implies that M is a finite extension of $\underline{\mathbb{F}}_p$-modules, from which we conclude. $\qquad\square$

6.4. **Motivic spaces.** Let k be a perfect field. Recall the category of motivic spaces $\mathcal{S}\mathrm{pc}(k)$ and motivic spectra $\mathcal{SH}(k)$, e.g. from [BH21b, §2.2 and §4.1]. Write $\Sigma^{p+q,q} \colon \mathcal{SH}(k) \to \mathcal{SH}(k)$ for the autoequivalence given by tensoring with the motivic sphere $\Sigma^p \mathbb{G}_m^q$. Pick $A \in \mathrm{CAlg}(\mathcal{SH}(k)^{\mathrm{veff}})$ (see Remark 6.27 for $\mathcal{SH}(k)^{\mathrm{veff}}$) and let $\mathcal{D} := \mathcal{M}\mathrm{od}_A(\mathcal{SH}(k))$. We will have to pick a category of solid objects, and there are some choices for this: for $\nu \geq 0$ put $\mathcal{D}_{\geq 0}^{\nu\mathrm{sld}} := \Sigma^{\nu,\nu} \mathcal{M}\mathrm{od}_A(\mathcal{SH}(k)^{\mathrm{veff}})$. For most purposes $\nu = 1$ is the only relevant case. For all $n \geq 0$ we have $\mathcal{X}_{*,\geq n}^{\nu\mathrm{sld}} \supseteq \mathcal{S}\mathrm{pc}(k)_{*,\geq(\nu+n,\nu)}$, cf. Remark 6.26.

Since $\mathcal{S}\mathrm{pc}(k)$ is famously *not* an ∞-topos, we cannot directly work with the adjunction $\mathcal{S}\mathrm{pc}(k) \rightleftarrows \mathcal{D}$, but have to work with $\mathcal{X} := \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$, the ∞-topos of Nisnevich sheaves on (quasi-compact) smooth k-schemes. This ∞-topos admits a locally finite-dimensional cover $\mathcal{U}$ [Mat24a, Proposition A.3]. Note that by construction there is an adjunction $L_{\mathbb{A}^1} \colon \mathcal{X} \rightleftarrows \mathcal{S}\mathrm{pc}(k) : \iota_{\mathbb{A}^1}$. Hence, we will work with the composed adjunction $L \colon \mathcal{X} \rightleftarrows \mathcal{S}\mathrm{pc}(k) \rightleftarrows \mathcal{D} : R$. We get the associated completion functor $\mathrm{Tot}\, T_A^\bullet \colon \mathcal{X}_* \to \mathcal{X}_*$ (which in fact takes values in $\mathcal{S}\mathrm{pc}(k)_* \subset \mathcal{X}$).

Remark 6.26. Recall the notion of ν-effective motivic spaces $\mathcal{S}\mathrm{pc}(k)_{\geq(\nu,\nu)}$ from [ABH24, §3], where the notation $O(S^{\nu,\nu})$ was used. It follows from the definition of $O(S^{d+\nu,\nu})$ in the stable situation together with [ABH24, Proposition 3.2.4] that if $X \in \mathcal{S}\mathrm{pc}(k)_*$ is ν-effective and d-connective, then $LX \in \mathcal{D}_{\geq d}^{\nu\mathrm{sld}}$.

Remark 6.27 (t-structures on $\mathcal{SH}(k)$). The stable ∞-category $\mathcal{SH}(k)$ admits multiple useful t-structures. The *homotopy t-structure* was defined by Morel [Mor03, Theorem 5.2.3], its connective and coconnective

parts are given by vanishing conditions on the bigraded homotopy sheaves. We will write $\mathcal{SH}(k)^\heartsuit$ for the heart of the homotopy t-structure, which Morel identified as the abelian category of homotopy modules [Mor03, Theorem 5.2.6]. As usual, we will write $\underline{\pi}_n(-)_*$ for the homotopy objects in this t-structure. This t-structure is left-complete, cf. [Hoy15, Corollary 2.4].

One other choice of a t-structure is the *effective t-structure*, whose connective part is given by $\mathcal{SH}(k)^{\mathrm{veff}}$, i.e., by those motivic spectra that are both connective in the homotopy t-structure and effective [Bac17, §3]. This t-structure restricts to a t-structure on $\mathcal{SH}(k)^{\mathrm{eff}}$, the subcategory of effective motivic spectra. We will write $\mathcal{SH}(k)^{\mathrm{eff}\heartsuit}$ for the heart, and $\underline{\pi}_n^{\mathrm{eff}}(-)$ for the homotopy objects in the effective t-structure.

Remark 6.28 (Slice filtration). For every $n \in \mathbb{Z}$ we write $f_n \colon \mathcal{SH}(k) \to \mathcal{SH}(k)$ for the n-th effective cover functor, and $s_n \coloneqq \mathrm{cof}(f_{n+1} \to f_n)$ for the n-th slice functor, cf. [Voe02, §2].

We will first verify the axioms.

Lemma 6.29. *The functors*

$$\mathcal{SH}(k)^{eff} \xrightarrow{\omega^\infty} \mathcal{SH}^{S^1}(k) \hookrightarrow \mathrm{Sp}(\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k))$$

preserve limits and colimits and are conservative.

Proof. Both functors are right adjoints by construction, and hence preserve limits. That ω^∞ is conservative and preserves colimits was shown in [BY20, Lemma 6.1 (1)]. The second functor is fully faithful and hence conservative. It is exact and thus preserves finite colimits. That it also preserves filtered colimits follows immediately from the fact that the Nisnevich sheaves of spectra $\Sigma_+^\infty U$ with $U \in \mathrm{Sm}_k$ are compact (see, e.g., [BH21b, Proposition A.3(2)]). $\square$

Lemma 6.30. *The axioms (C), (M), (S1) and (S3) hold for the adjunction $L \dashv R$. If $\nu \geq 1$ and $\underline{\pi}_0(A)_*$ is idempotent in $\mathcal{SH}(k)^\heartsuit$, then also (S2) holds.*

Proof. For axioms (C) and (S3) we consider the following commutative diagram:

$$
\begin{array}{ccccccc}
\mathrm{Sp}(\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k))_{\geq 0} & \longleftarrow & \mathcal{SH}^{S^1}(k)_{\geq 0} & \xleftarrow{\omega^\infty} & \mathcal{SH}(k)^{\mathrm{veff}} & \longleftarrow & \mathcal{M}\mathrm{od}_A(\mathcal{SH}(k)^{\mathrm{veff}}) \\
\downarrow{\scriptstyle\Omega^\infty} & & \downarrow & & & & \\
\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k) & \longleftarrow & \mathcal{S}\mathrm{pc}(k). & & & &
\end{array}
$$

The functor R is given by the composition from the top right to the bottom left. Hence, it suffices to show that any functor in this composition preserves sifted colimits and is conservative. For the forgetful functor $\mathcal{M}\mathrm{od}_A(\mathcal{SH}(k)^{\mathrm{veff}}) \to \mathcal{SH}(k)^{\mathrm{veff}}$, this follows from e.g. [Lur17, Proposition 4.8.5.8 (4) and Corollary 4.2.3.7 (2)]. See Lemma 6.29 for ω^∞ and the inclusion. In the proof of Lemma 6.5 we showed the statement for Ω^∞.

For axiom (M), it suffices to note that for any $X \in \mathcal{X}_*$, we have $LX \in \mathcal{D}_{\geq 0}^{0\mathrm{sld}}$ (see Remark 6.26), and that

$$D_{\geq b}^{a\mathrm{sld}} \otimes D_{\geq b'}^{a'\mathrm{sld}} \subset D_{\geq b+b'}^{(a+a')\mathrm{sld}}.$$

The same reasoning also implies axiom (S1), using that Ω^∞ preserves effectivity [ABH24, Proposition 3.2.12].

For (S2), let us first show that if $E \in \Sigma^{2,1}\mathcal{SH}(k)^{\mathrm{veff}}$ then $\Sigma^\infty\Omega^\infty E \to E$ induces an isomorphism on $\underline{\pi}_1(-)_*$. Equivalently, using [Bac17, Proposition 5], the composite

$$\Sigma^{2,1}\mathcal{SH}(k)^{\mathrm{eff}\heartsuit} \to \Sigma\mathcal{SH}^{S^1}(k)(1)^\heartsuit \to \mathcal{X}_*$$

should be fully faithful. The first functor is fully faithful by [BY20, Theorem 6.9], and the second by the classical Hurewicz theorem together with [ABH24, Proposition 3.2.12]. Now let $E \in \mathcal{D}_{\geq 1}^{\nu\mathrm{sld}}$. The functor $\mathcal{D}^{0\mathrm{sld}} \to \mathcal{SH}(k)^{\mathrm{eff}}$ being t-exact (see e.g. [Bac21, Lemma 29]), in order to show that $LRE \to E$ induces an isomorphism on π_1 (with respect to the t-structure specified by $\mathcal{D}^{\nu\mathrm{sld}}$), it is by [Bac17, Proposition 4] sufficient (and necessary) to show that $\underline{\pi}_1(LRE)_{-\nu} \simeq \underline{\pi}_1(E)_{-\nu}$. Since $\nu \geq 1$ we have (by what we first showed) that $\underline{\pi}_1(LRE)_* \simeq \underline{\pi}_1(E)_* \otimes \underline{\pi}_0(A)_*$. This implies what we want since $\underline{\pi}_0(A)_*$ is idempotent by assumption. $\square$

In the cases discussed below, $\underline{\pi}_0(A)_*$ will always be either a quotient or a localization of the unit (or a combination of the two), whence idempotent, so that the axioms hold.

Corollary 6.31. *Let $X \in \mathcal{S}\mathrm{pc}(k)_*$ be n-effective and d-connected. Then $T_A^\bullet(X)$ consists of n-effective d-connected spaces.*

Proof. This follows from Corollary 5.4 by taking $\nu = n$. □

Before stating the main theorems of this section, we need some preliminaries.

Definition 6.32. We call $X \in \mathcal{S}\mathrm{pc}(k)_*$ *resolvable* if there exists a tower $X \to X_\bullet$ as well as fiber sequences $X_{i+1} \to X_i \to \Omega^\infty B_i$, such that:

  (1) The connectivity of $X \to X_i$ tends to infinity with i.
  (2) The connectivity of B_i tends to infinity with i.
  (3) $B_i \in \Sigma^{3,1}\mathcal{S}\mathcal{H}(k)^{\mathrm{veff}}$.
  (4) $X_0 = *$.

Remark 6.33. It is proved in [ABH24, Construction 4.1.7, Remark 4.1.13] that $X \in \mathcal{S}\mathrm{pc}(k)_*$ is resolvable as soon as X is nilpotent and either

  • X is 2-effective, or
  • X is 1-effective and e-periodic, where e is the exponential characteristic of k.

Remark 6.34. It follows that $X_i \in \mathcal{X}^{\mathrm{1sld}}_{*,\geq 1}$ and $\Omega^\infty B_i \in \mathcal{X}^{\mathrm{1sld}}_{*,\geq d_i}$ for some sequence d_i tending to infinity with i. Indeed, the second claim is (essentially) Lemma 3.4 (2), combined with the assumption that the connectivity of the B_i tends to infinity with i. The first claim is proven inductively, using the second claim, [ABH24, Theorem 4.2.3] and the definition of $\mathcal{D}^{\mathrm{1sld}}$.

Remark 6.35. If X is resolvable and n-effective, one may choose the X_i and B_i such that they are all n-effective, again by [ABH24, Construction 4.1.7 and Remark 4.1.13].

The next result concerns the interaction between the ∞-topos-theoretic S-localization in $\mathcal{X}$ and the stable S-localization in $\mathcal{S}\mathcal{H}(k)$.

Lemma 6.36. *Let $S \subset \mathbb{Z}$, and let $E \in \mathcal{S}\mathcal{H}(k)$ be 1-connective. Denote by $L_S \colon \mathcal{X} \to \mathcal{X}$ the unstable S-localization functor. Then $L_S\Omega^\infty E \simeq \Omega^\infty S^{-1}E$.*

Proof. Consider the commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Sp}(\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)) & \longleftarrow & \mathcal{S}\mathcal{H}^{S^1}(k) & \overset{\omega^\infty}{\longleftarrow} & \mathcal{S}\mathcal{H}(k)^{\mathrm{eff}} \\
\downarrow{\scriptstyle\Omega^\infty} & & \downarrow & & \\
\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k) & \longleftarrow & \mathcal{S}\mathrm{pc}(k). & &
\end{array}
$$

It suffices to show that the composition on the top and left commutes with S-localization. Since both ω^∞ and the inclusion $\mathcal{S}\mathcal{H}^{S^1}(k) \hookrightarrow \mathrm{Sp}(\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k))$ preserve colimits by Lemma 6.29, it follows from e.g. [Mat24a, Corollary 2.7] that they commute with S-localization. Moreover, both of these functors preserve 1-connective objects. Hence, the result follows from [Mat24a, Lemma 3.18]. □

We also need a version of the last lemma for the n-completion functors:

Lemma 6.37. *Let $n \in \mathbb{Z}$, and let $E \in \mathcal{S}\mathcal{H}(k)$ be 1-connective. Denote by $L_n \colon \mathcal{X} \to \mathcal{X}$ the unstable n-completion functor from Definition C.2. Then $L_n\Omega^\infty E \simeq \tau_{\geq 1}\Omega^\infty E_n^\wedge$.*

Proof. Since $\Omega^\infty E$ is nilpotent (as E is 1-connective), we have that $L_n\Omega^\infty E \simeq \prod_{\ell | n} L_\ell\Omega^\infty E$, cf. Theorem C.6. Similarly, from Lemma C.1 we have $E_n^\wedge \simeq \prod_{\ell | n} E_\ell^\wedge$. Since Ω^∞ and $\tau_{\geq 1}$ preserve finite products, we may assume that $n = \ell$. Consider again the commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Sp}(\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)) & \longleftarrow & \mathcal{S}\mathcal{H}^{S^1}(k) & \overset{\omega^\infty}{\longleftarrow} & \mathcal{S}\mathcal{H}(k)^{\mathrm{eff}} \\
\downarrow{\scriptstyle\Omega^\infty} & & \downarrow & & \\
\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k) & \longleftarrow & \mathcal{S}\mathrm{pc}(k). & &
\end{array}
$$

It suffices to show that the composition on the top and left commutes with ℓ-completion, up to a connected cover. The functors on the top both preserve limits by Lemma 6.29, and hence commute with ℓ-completion, see e.g. [Mat24b, Lemma 2.32]. Moreover, they preserve 1-connective objects. Hence, the result follows from [Mat24b, Lemma 3.17]. □

We now discuss Bousfield–Kan completions. We now have four different situations: First, we can look at those A such that $\underline{\pi}_0(A)_* = S^{-1}\underline{\pi}_0(\mathbb{1})_*$ or $\underline{\pi}_0(A)_* = \underline{\pi}_0(\mathbb{1})_*/n$; this is similar to the case of sheaves discussed above. But these two cases do not cover two important situations: One wants to be able to complete at $A = H\mathbb{Z}$, the motivic cohomology spectrum, or $A = \mathrm{MGL}$, the algebraic cobordism

spectrum. Here, we do not have that $\underline{\pi}_0(A)_* = \underline{\pi}_0(\mathbb{1})_*$, but instead $\underline{\pi}_0(A)_* = \underline{\pi}_0(\mathbb{1})_*/\eta$. Thus, in the second half of this section, we will look at completions at those A such that either $\underline{\pi}_0(A)_* = S^{-1}\underline{\pi}_0(\mathbb{1})_*/\eta$, or $\underline{\pi}_0(A)_* = \underline{\pi}_0(\mathbb{1})_*/(\eta, n)$. This will complicate things slightly, as it will also include an additional η-completion. To circumvent this problem, we will restrict ourselves to fields of finite 2-étale cohomological dimension, so that all very effective motivic spectra are already η-complete.

6.4.1. *Milnor–Witt Resolutions: Localization.* In this subsection, we assume that $\underline{\pi}_0(A)_* \simeq S^{-1}\underline{\pi}_0(\mathbb{1})_*$ with $S \subset \mathbb{Z} \setminus 0$. Thus, $\underline{\pi}_0(A)_* \otimes_{\underline{\pi}_0(\mathbb{1})_*} \underline{\pi}_0(A)_* \simeq \underline{\pi}_0(A)_*$, and hence all axioms are satisfied. In this situation, the Bousfield–Kan completion is related to unstable S-localization. We write L_S for the unstable S-localization functor in $\mathcal{X}$, cf. [Mat24a, §3].

Example 6.38. Possible examples for A are the rings $S^{-1}\mathbb{1}$ and $f_0 S^{-1}\underline{K}^{MW}$.

Theorem 6.39. (1) *Let* $X \in \mathcal{S}\mathrm{pc}(k)_*$ *be nilpotent. Then* $L_S X \in \mathcal{S}\mathrm{pc}(k)_*$ *(i.e.,* $L_S X$ *is again* $\mathbb{A}^1$*-invariant).*

(2) *Let* $X \in \mathcal{S}\mathrm{pc}(k)_*$ *be resolvable. The canonical maps* $\tau_{<\infty} L_S X \to \tau_{<\infty} \hat{L}_A X \to \tau_{<\infty} T^\bullet X$ *are equivalences. In particular,* $L_S X \to \hat{L}_A X \to \mathrm{Tot}\, T^\bullet X$ *are equivalences.*

(3) *For* $E \in \mathcal{SH}(k)_{\geq 1}$ *the canonical map* $L_S \Omega^\infty E \to \Omega^\infty L_A^{st} E$ *is an equivalence.*

(4) *Let* $F \to E \to B$ *be a fiber sequence of pointed nilpotent Nisnevich sheaves. Then* $L_S F \to L_S E \to L_S B$ *is a fiber sequence.*

(5) *Let* $X_\bullet$ *be a tower of pointed nilpotent Nisnevich sheaves with* $X = \lim_i X_i$. *Assume that the tower is locally highly connected subordinate to* $\mathcal{U}$, *e.g. it is highly connected. Then* $L_S X \simeq \lim_i L_S X_i$.

Proof. Statement (1) is proven in [AFH22, Proposition 4.3.8 and Theorem 4.3.9], statement (4) is [Mat24a, Lemma 3.13], and statement (5) is [Mat24a, Proposition 6.9].

For (3), since $\Omega^\infty E$ is nilpotent, we have

$$L_S \Omega^\infty E \simeq \Omega^\infty S^{-1} E \simeq \Omega^\infty L_A^{st} E,$$

where we used Lemma 6.36 for the first, and Proposition B.1 for the second equivalence.

We now prove (2). Note that L inverts S-equivalences, and so in order to show that $L_S X \simeq \hat{L}_A X$, it suffices to show that $L_S X$ is L-local. This is clearly true for $\mathrm{Tot}\, T^\bullet X$, and so, it is enough to show that $\tau_{<\infty} L_S X \simeq \tau_{<\infty} T^\bullet X$. Write $F \coloneqq T^\bullet$. Let X be resolvable, and consider the generalized Postnikov tower. The functor L_S preserves all fiber sequences and the limit appearing in the tower by (4) and (5), and similarly F preserves all those fiber sequences by Theorem 5.7. To see that F preserves the limit, note that $(X_i)_i$ is a highly connected tower (under X), and we conclude by Lemma 6.7 that the connectivity of $X \to X_i$ tends to ∞. Thus, by Proposition 5.3 also the connectivity of $FX \to FX_i$ tends to ∞, even uniformly in the levels of the pro-object. Hence, by definition of $\tau_{<\infty}$, it follows that $\tau_{<\infty} FX \simeq \tau_{<\infty} \lim_i FX_i$. Hence, we reduced to the case that $X = \Omega^\infty E$ for some motivic spectrum $E \in \Sigma^{3,1} \mathcal{SH}(k)^{\mathrm{veff}}$. Consider now the Postnikov tower of E in the 1-effective t-structure on $\mathcal{SH}(k)(1)$. With exactly the same arguments as above, we can thus further reduce to the case that $E \in \Sigma^{m+1,1} \mathcal{SH}(k)^{\mathrm{eff}\heartsuit}$ for $m \geq 2$. Since both sides invert L_S-equivalences (for F note that already L inverts them, since any $m \in S$ is invertible on A, for this note that it is enough to check that m is invertible on $\underline{\pi}_*(A)_*$, but $\underline{\pi}_*(A)_*$ is a $\underline{\pi}_0(A)_*$-algebra, and m is invertible on $\underline{\pi}_0(A)_* = S^{-1}\underline{\pi}_0(\mathbb{1})_*$), we can replace X by $L_S X$, and we have to see that $\tau_{<\infty} X \to \tau_{<\infty} FX$ is an equivalence. By (3) we know $X = L_S \Omega^\infty E \simeq \Omega^\infty S^{-1} E$. Note that $S^{-1} E$ has the structure of a $S^{-1}\underline{\pi}_0^{\mathrm{eff}}(\mathbb{1})$-module. We have

$$S^{-1} \underline{\pi}_0^{\mathrm{eff}}(\mathbb{1}) \simeq S^{-1} f_0 \underline{\pi}_0(\mathbb{1})_* \simeq f_0 S^{-1} \underline{\pi}_0(\mathbb{1})_* \simeq f_0 \underline{\pi}_0(A)_* \simeq \underline{\pi}_0^{\mathrm{eff}}(A),$$

where we used that $\mathbb{1}$ and A are effective, that f_0 preserves colimits, and that f_0 is t-exact from the homotopy t-structure to the effective t-structure, see [BY20, Lemma 6.2 (2)]. In particular, $S^{-1} E$ acquires the structure of a $\underline{\pi}_0^{\mathrm{eff}}(A)$-module, and hence also has the structure of an A-module (via the canonical ring map $A \to \underline{\pi}_0^{\mathrm{eff}}(A)$). Hence, X is in the essential image of R, and hence $\tau_{<\infty} X \to \tau_{<\infty} FX$ is an equivalence by Proposition A.4 (even before applying $\tau_{<\infty}$). $\square$

6.4.2. *Milnor–Witt Resolutions: Completion.* We assume now that $\underline{\pi}_0(A)_* \simeq \underline{\pi}_0(\mathbb{1})_*/n$ with $n \in \mathbb{N}$. Thus, $\underline{\pi}_0(A)_* \otimes_{\underline{\pi}_0(\mathbb{1})_*} \underline{\pi}_0(A)_* \simeq \underline{\pi}_0(A)_*$, and hence all axioms are satisfied. In this situation, the Bousfield–Kan completion is related to unstable n-completion. We write L_n for the unstable n-completion functors in $\mathcal{X}$, cf. Definition C.2.

Example 6.40. A possible example for A is the ring $f_0(\underline{K}^{MW}/n)$.

Theorem 6.41. (1) *Let* $X \in \mathcal{S}\mathrm{pc}(k)_*$ *be nilpotent. Then* $L_n X \in \mathcal{S}\mathrm{pc}(k)_*$ *(i.e.,* $L_n X$ *is again* $\mathbb{A}^1$*-invariant).*

(2) Let $X \in \mathcal{S}\mathrm{pc}(k)_*$ be resolvable. There is a canonical map $L_n X \to \tau_{\geq 1} \operatorname{Tot} T^\bullet X$, which is an equivalence.

(3) For $E \in \mathcal{SH}(k)_{\geq 1}$ the canonical map $L_n \Omega^\infty E \to \tau_{\geq 1} \Omega^\infty L_A^{st} E$ is an equivalence.

(4) Let $F \to E \to B$ be a fiber sequence of pointed nilpotent Nisnevich sheaves. Then the canonical map $L_n F \to \tau_{\geq 1} \operatorname{fib}(L_n E \to L_n B)$ is an equivalence.

(5) Let $X_\bullet$ be a highly connected tower of pointed nilpotent Nisnevich sheaves with $X = \lim_i X_i$. Then the canonical map $L_n X \to \lim_i L_n X_i$ is an equivalence.

Proof. For (1) note that $L_n X \simeq \prod_{\ell|n} L_\ell X$ by Theorem C.6. Thus, since $\mathbb{A}^1$-invariant sheaves are stable under limits, the claim follows from [Mat24b, Proposition 5.22]. Moreover, (3) is Lemma 6.37 combined with Proposition B.1, (4) is Lemma C.17 and (5) is Lemma C.18.

We now show (2). Since $\tau_{\geq 1} \colon \mathcal{X}_* \to \mathcal{X}_{*,\geq 1}$ is a right adjoint, it preserves limits. Similarly, the functor L_n restricts to $\mathcal{X}_{*,\geq 1}$ (this follows from [Mat24b, Lemma 3.12]), and the restricted functor preserves appropriate fiber sequences (by Lemma C.17). As in the proof for $\underline{\pi}_0(A)_* = S^{-1}\underline{\pi}_0(\mathbb{1})_*$, we reduce to the case that $X = \Omega^\infty \Sigma^k M$ for $M \in \Sigma^{1,1} \mathcal{SH}(k)^{\mathrm{eff}\heartsuit}$ with $k \geq 2$. Write $F \coloneqq \tau_{\geq 1} \operatorname{Tot} T_A^\bullet = \tau_{\geq 1} \operatorname{mat} \tau_{<\infty} T_A^\bullet$. Then

$$G(M) \coloneqq (F\Omega^\infty \Sigma^k M, F\Omega^\infty \Sigma^{k+1} M, \dots) \in \operatorname{Sp}(\mathcal{S}\mathrm{pc}(k)_{*,\geq 1}) \simeq \mathcal{SH}^{S^1}(k),$$

since F preserves loops by Theorem 5.7. The functor G also preserves fiber sequences of 2-connective spectra. There is a canonical map $\omega^\infty \Sigma^k M \to G(M)$. Assume for now that this is an equivalence whenever M is a finite extension of $\underline{\pi}_0^{\mathrm{eff}}(\mathbb{1})/n$-modules. It follows that $\Sigma^k M \to G(M)$ is an n-equivalence, i.e., induces an equivalence on fibers of multiplication by n. Since $\omega^\infty \Sigma^k M$ is connected, this implies that also $\omega^\infty \Sigma^k M \to \tau_{\geq 1} G(M)$ is an n-equivalence (this follows easily from e.g. [Mat24b, Lemma 2.9 and Corollary 2.11]; note that in the proof there it is never used that p is a prime). Hence, using Lemma C.16, we see that

$$\Omega^\infty \Sigma^k M \to \Omega_{S^1}^\infty \tau_{\geq 1} G(M) \simeq F(\Omega^\infty \Sigma^k M)$$

is also an n-equivalence. Since the target is n-complete by construction, we conclude.

We end the proof by showing that $\Sigma^k M \to G(M)$ is an equivalence if M is a finite extension of $\underline{\pi}_0^{\mathrm{eff}}(\mathbb{1})/n$-modules. By preservation of fiber sequences, we immediately reduce to the case where M itself is such a module. For this, we will show that in fact M is even an A-module, whence the claim follows from Proposition A.4. We have

$$\underline{\pi}_0^{\mathrm{eff}}(\mathbb{1})/n \simeq f_0(\underline{\pi}_0(\mathbb{1})_*/n) \simeq f_0 \underline{\pi}_0(A)_* \simeq \underline{\pi}_0^{\mathrm{eff}}(A),$$

where we used that $\mathbb{1}$ and A are effective, and that f_0 is t-exact from the homotopy t-structure to the effective t-structure, see [BY20, Lemma 6.2 (2)]. Thus, M acquires the structure of a $\underline{\pi}_0^{\mathrm{eff}}(A)$-module, and hence also has the structure of an A-module (via the canonical ring map $A \to \underline{\pi}_0^{\mathrm{eff}}(A)$). This finishes the proof. $\square$

6.4.3. *Milnor Resolutions: Generalities.* From now on, we will assume that $\operatorname{cd}_2(k) < \infty$. We will now study the situation that $\underline{\pi}_0(A)_* \simeq S^{-1}(\underline{\pi}_0(\mathbb{1})_*/\eta)$, or that $\underline{\pi}_0(A)_* \simeq \underline{\pi}_0(\mathbb{1})_*/(\eta, n)$.

Remark 6.42. We quickly explain why we need the assumption that k has finite 2-étale cohomological dimension. In topology, going from the sphere $\mathbb{S}_{\mathrm{top}}$ to its 0-truncation $\pi_0(\mathbb{S}_{\mathrm{top}})$ kills off only nilpotent elements, hence the associated completion functors agree (at least on bounded below objects). In contrast, going from the motivic sphere $\mathbb{1}$ to $\underline{\pi}_0(\mathbb{1})_*/\eta$ kills the motivic Hopf map $\eta \colon \mathbb{G}_m \to \mathbb{1}$, which is known to be non-nilpotent [Mor04, Corollary 6.4.5]. Hence, homological localization with respect to $H\mathbb{Z}$ introduces a completion operation with respect to η. Unstably, the associated Bousfield localization is currently poorly understood. The situation gets much better if one requires the 2-étale cohomological dimension to be finite, as then every very effective motivic spectrum is automatically η-complete, and hence the associated unstable Bousfield localization is just the identity.

Lemma 6.43. *Assume that $\operatorname{cd}_2(k) < \infty$, and let $E \in \mathcal{SH}(k)^{veff}$ be a very effective motivic spectrum. Then E is η-complete. If E is 2-power-torsion, then E is slice-complete.*

Proof. Since $\operatorname{cd}_2(k) < \infty$ we know that $E[1/2]$ is η-complete. Indeed, there is a canonical splitting $E[1/2] \simeq E[1/2]^+ \oplus E[1/2]^-$ with $E[1/2]^+$ an η-complete motivic spectrum by e.g. [BH21a, §2.7.3], and under our assumption on the cohomological dimension, we have $E[1/2]^- = 0$: Indeed, the splitting of $E[1/2]$ is induced by a splitting of the whole ∞-category $\mathcal{SH}(k)[1/2]$, since there is a splitting of the endomorphisms of the unit $GW(k)[1/2] \simeq \mathbb{Z}[1/2] \times W(k)[1/2]$. Hence, to show that $E[1/2]^- = 0$, it suffices to show that the Witt ring $W(k)$ has 2^N torsion for some $N \gg 0$. Write $I \subset W(k)$ for the fundamental ideal. Then, by [MH+73, the discussion after Theorem 5.1], all the quotients I^n/I^{n+1} are

$\mathbb{F}_2$-vectorspaces, hence 2-torsion. It thus is enough to show that $I^N = 0$ for $N \gg 0$, for this one can argue as in the proof of [Bac18b, Theorem 16], using that $\mathrm{cd}_2(k) < \infty$.

Using that η-complete spectra are stable under fiber sequences, replacing E by the cofiber of $E \to E[1/2]$ we may assume that E is 2-power-torsion. To prove that E is η-complete it now suffices to show that it is slice-complete (indeed, since E is effective, the slice completion is a limit of finite extensions of slices, and slices are η-complete as $s_0(\mathbb{1}) = H\mathbb{Z}$). This follows from Levine's convergence theorem. If k has characteristic $\neq 2$, see [Lev13, Theorem 7.3] (also recalled in [BEØ24, Theorem 5.3]). If k has characteristic 2, see our variant in Corollary D.2. $\qquad\square$

We need the following ad-hoc definition:

Definition 6.44. Let $X \in \mathcal{S}\mathrm{pc}(k)_*$ be resolvable. We say that X is 2-*power-torsion* if one can choose the X_i and B_i appearing in Definition 6.32 so that B_i is 2-power-torsion (i.e. $B_i[1/2] = 0$) for all i.

Lemma 6.45. *Suppose that $\mathrm{cd}_2(k) < \infty$, and let $X \in \mathrm{Pro}(\mathcal{S}\mathrm{pc}(k)_*)$ be nilpotent, 2-power-torsion and ∞-effective, i.e., $X \simeq \lim_i cX_i$, where X_i is nilpotent, 2-power-torsion and n_i-effective, with $\lim_i n_i = \infty$. Then $\mathrm{mat}\, X = *$.*

Proof. It suffices to show that if $X \in \mathcal{S}\mathrm{pc}(k)_*$ is nilpotent, 2-power-torsion and n-effective (without loss of generality we may assume that $n \geq 2$, so that X is resolvable by Remark 6.33) and $U \in \mathrm{Sm}_k$ has dimension $\leq d$, then $X|_{U_{\mathrm{Nis}}}$ is $N(d,n)$-connective, where $N(d,n)$ is a function (independent of X and U) with $\lim_n N(d,n) = \infty$. (Indeed, the restrictions to the small sites U_{Nis} preserve all limits and are jointly conservative, and moreover $\lim_i(X_i|_{U_{\mathrm{Nis}}}) = *$ by e.g. [Mat25, Lemma 2.4] and the increasing connectivity of the $X_i|_{U_{\mathrm{Nis}}}$.) By an induction on the generalized Postnikov tower of Definition 6.32, we may assume that $X = \Omega^\infty E$ for some $E \in \Sigma^{n,n}\mathcal{SH}(k)^{\mathrm{veff}}$ with $E[1/2] = 0$. The claim now follows from Levine's convergence theorem; see [Lev13, Theorem 7.3] (also recalled in [BEØ24, Theorem 5.3]) if k has characteristic $\neq 2$, and Corollary D.2 else. $\qquad\square$

6.4.4. *Milnor Resolutions: Localization.* In this subsection, we assume that $\underline{\pi}_0(A)_* \simeq S^{-1}(\underline{\pi}_0(\mathbb{1})_*/\eta)$ with $S \subset \mathbb{Z} \setminus 0$. In this situation, the Bousfield–Kan completion is related to unstable S-localization. We write L_S for the unstable S-localization functor in $\mathcal{X}$, cf. [Mat24a, §3].

Example 6.46. Possible examples for A are the rings $S^{-1}H\mathbb{Z}$ and $S^{-1}\mathrm{MGL}$. Of course, we may take $S = \{1\}$.

Before stating the main theorem in this situation, let us identify the stable localization functor.

Lemma 6.47. *Assume that $\mathrm{cd}_2(k) < \infty$. Let $E \in \mathcal{SH}(k)^{\mathrm{veff}}$. Then $L_A^{st}(E) \simeq S^{-1}E$.*

Proof. Let $m \in S$. We first show that $A \otimes (E \xrightarrow{m} E)$ is invertible, for which it suffices to show that m is invertible on A, which holds since m is invertible on $\underline{\pi}_0(A)_* = S^{-1}\underline{\pi}_0(\mathbb{1})_*/\eta$. Thus, we see that the canonical map $E \to L_A^{st}(E)$ factors through $E \to S^{-1}E$, and that the resulting map $S^{-1}E \to E$ is inverted by $L_A^{st}(-)$. Hence, it suffices to show that $S^{-1}E$ is L_A^{st}-local. We have morphisms $S^{-1}\mathbb{1} \to A \to \underline{\pi}_0(A)_* = S^{-1}(\underline{\pi}_0(\mathbb{1})_*/\eta)$, and hence, get a diagram

$$L_{S^{-1}\mathbb{1}}^{st}E \to L_A^{st}E \to L_{S^{-1}(\underline{\pi}_0(\mathbb{1})_*/\eta)}^{st}E.$$

By Proposition B.1 we see that

$$L_{S^{-1}\mathbb{1}}^{st}E \simeq S^{-1}E$$

and

$$L_{S^{-1}(\underline{\pi}_0(\mathbb{1})_*/\eta)}^{st}E \simeq L_{S^{-1}\mathbb{1}/\eta}^{st}E \simeq (S^{-1}E)_\eta^\wedge \simeq S^{-1}E,$$

where the last equivalence is Lemma 6.43. Thus, the above sequence of morphisms is equivalent to a retract diagram $S^{-1}E \to L_A^{st}E \to S^{-1}E$. Since $L_A^{st}E$ is L_A^{st}-local, the same is true for the retract $S^{-1}E$, which is what we wanted to show. $\qquad\square$

Theorem 6.48. *Assume that $\mathrm{cd}_2(k) < \infty$.*
   (1) *Let $X \in \mathcal{S}\mathrm{pc}(k)_*$ be nilpotent. Then $L_S X \in \mathcal{S}\mathrm{pc}(k)_*$ (i.e., $L_S X$ is again $\mathbb{A}^1$-invariant).*
   (2) *Let $X \in \mathcal{S}\mathrm{pc}(k)_*$ be resolvable. The canonical maps $L_S X \to \hat{L}_A X \to \mathrm{Tot}\, T_A^\bullet X$ are equivalences.*
   (3) *For $E \in \mathcal{SH}(k)_{\geq 1}^{\mathrm{eff}}$ the canonical map $L_S \Omega^\infty E \to \Omega^\infty L_A^{st} E$ is an equivalence.*
   (4) *Let $F \to E \to B$ be a fiber sequence of pointed nilpotent Nisnevich sheaves. Then $L_S F \to L_S E \to L_S B$ is a fiber sequence.*
   (5) *Let $X_\bullet$ be a tower of pointed nilpotent Nisnevich sheaves with $X = \lim_i X_i$. Assume that the tower is locally highly connected subordinate to $\mathcal{U}$, e.g. it is highly connected. Then $L_S X \simeq \lim_i L_S X_i$.*

Proof. (1), (4) and (5) were already shown in Theorem 6.39.

For (3) we combine Lemma 6.36 and Lemma 6.47.

Hence, the only thing left is (2). As before, since L inverts S-equivalences, in order to prove that $L_S X \simeq \hat{L}_A X$, it suffices to show that $L_S X \simeq \mathrm{Tot}\, T_A^\bullet X$ (the latter being L-local). Write $F := \mathrm{Tot}\, T_A^\bullet$. It is clear that F inverts S-equivalences and hence the canonical map $X \to FX$ factors through $L_S X$. Let X be resolvable, and consider its generalized Postnikov tower (Definition 6.32). The functor L_S preserves all fiber sequences and the limit appearing in the tower by (4) and (5), and similarly F preserves all those fiber sequences by Theorem 5.7 and Remark 6.34. To see that F preserves the limit, note that $(X_i)_i$ is a highly connected tower (under X), and we conclude by Lemma 6.7 that the connectivity of $X \to X_i$ tends to ∞. Thus, by Proposition 5.3 also the connectivity of $FX \to FX_i$ tends to ∞. Moreover, the same proposition shows that FX_i is a highly connected tower. In particular, it follows again from Lemma 6.7 that $FX \simeq \lim_i FX_i$. We have thus reduced to the case $X = \Omega^\infty E$ for some motivic spectrum $E \in \Sigma^{3,1} \mathcal{SH}(k)^{\mathrm{veff}}$. In fact, since both sides invert S-equivalences, we may assume that $E \simeq S^{-1}E$, and we must show that $X \to FX$ is an equivalence. Consider the fiber sequence

$$E \to E[1/2] \to C.$$

Applying Ω^∞, we obtain a fiber sequence preserved by F (by Theorem 5.7, as all involved motivic spaces are 1-effective and 2-connective) and L_S; it thus suffices to treat $E[1/2]$ and C, i.e., 2-periodic spectra and 2-power-torsion spectra.

Let E be 2-periodic. Then since $\mathrm{cd}_2(k) < \infty$, η acts by zero on E: Indeed, as in the proof of Lemma 6.43, we see that $E \simeq E^+$ (i.e. the minus-part vanishes), and we know from [Bac18a, Lemma 39] that η acts by zero on E^+. Consider now the Postnikov tower of E in the 1-effective t-structure on $\mathcal{SH}(k)(1)$. By an analogous argument as in the beginning of the proof we are thus allowed to reduce to the case where $E \in \Sigma^{k+1,1} \mathcal{SH}(k)^{\mathrm{eff}\heartsuit}$ for some $k \geq 2$, which is still S- and 2-periodic. Consider $\Sigma^k \underline{\pi}_k(E)_*$, which again is S- and 2-periodic, and hence η acts by zero on E. In particular, $\Sigma^k \underline{\pi}_k(E)_*$ is a $(S^{-1} \underline{\pi}_0(\mathbb{1})_*/\eta = \underline{\pi}_0(A)_*)$-module. Applying f_1, which is t-exact from the homotopy t-structure to the 1-effective t-structure by [BY20, Lemma 6.2 (2)], we get that $E = \Sigma^k \underline{\pi}_k^{1\mathrm{eff}}(E) \simeq \Sigma^k f_1 \underline{\pi}_k(E)_* \simeq \Sigma^{k,1} f_0 \Sigma^{0,-1} \underline{\pi}_k(E)_*$ is an $(f_0 \underline{\pi}_0(A)_* = \underline{\pi}_0^{\mathrm{eff}}(A))$-module. In particular, it is also an A-module, and thus the canonical map $\Omega^\infty E \to F \Omega^\infty E$ is an equivalence by Proposition A.4.

Now let E be 2-power-torsion. Consider the fiber sequences $f_n E \to E \to E/f_n E$, which are preserved by $F \Omega^\infty$. We have $F \Omega^\infty(E/f_n E) \simeq \Omega^\infty(E/f_n E)$: Indeed, $E/f_n E$ is a finite extension of slices (since E is effective). Thus, as above we can reduce to the case of $s_n(E)$. Slices are $H\mathbb{Z}$-modules, hence $s_n(E)$ is an $S^{-1} H\mathbb{Z}$-module. We have

$$S^{-1} H\mathbb{Z} \simeq S^{-1} f_0(\underline{\pi}_0(\mathbb{1})_*/\eta) \simeq f_0 \underline{\pi}_0(A)_* \simeq \pi_0^{\mathrm{eff}}(A),$$

using [BY20, Lemma 6.2 (2)] and [Bac17, Lemma 12]. In particular, we see that $s_n(E)$ is an $\pi_0^{\mathrm{eff}}(A)$-module and hence an A-module, and thus $F \Omega^\infty(s_n E) \simeq \Omega^\infty(s_n E)$ by Proposition A.4. Taking the limit over n we obtain a fiber sequence

$$\mathrm{mat}(\lim_{n,m} c\, \mathrm{Tot}_m T_A^\bullet \Omega^\infty(f_n E)) \to F \Omega^\infty E \to \lim_n \Omega^\infty E/f_n E \simeq \Omega^\infty E,$$

where the equivalence is by Lemma 6.43. We finish the proof by showing that the second map is an equivalence. Since the base is connected, for this it will be enough to show that the fiber is contractible. This follows from Corollary 6.31 and Lemma 6.45. $\qquad\square$

6.4.5. *Milnor Resolutions: Completion.* In this subsection, we assume that $\underline{\pi}_0(A)_* \simeq \underline{\pi}_0(\mathbb{1})_*/(\eta, n)$ with $n \in \mathbb{N}$. In this situation, the Bousfield–Kan completion is related to unstable n-completion. We write L_n for the unstable n-completion functor in $\mathcal{X}$, cf. Definition C.2.

Example 6.49. A possible example for A is the ring $H\mathbb{Z}/n$.

Before stating the main theorem in this situation, let us identify the stable localization functor.

Lemma 6.50. *Assume that* $\mathrm{cd}_2(k) < \infty$. *Let* $E \in \mathcal{SH}(k)^{\mathrm{veff}}$. *Then* $L_A^{st}(E) \simeq E_n^\wedge$.

Proof. By assumption, we get morphisms $\mathbb{1}/n \to A \to \pi_0^{\mathrm{eff}}(A) = H\mathbb{Z}/n$, and thus a diagram

$$L_{\mathbb{1}/n}^{st} E \to L_A^{st} E \to L_{H\mathbb{Z}/n}^{st} E,$$

where the first map is inverted by $L_A^{st}(-)$. By definition $L_{\mathbb{1}/n}^{st} E = E_n^\wedge$. Hence, it suffices to show that $L_{\mathbb{1}/n}^{st} E$ is L_A^{st}-local. By [Man24, §5.2] we get

$$L_{H\mathbb{Z}/n}^{st} E \simeq L_{\mathbb{1}/(\eta,n)}^{st} E = E_{\eta,n}^\wedge \simeq E_n^\wedge,$$

where the last equivalence is Lemma 6.43. Thus, the above sequence of morphisms is equivalent to a retract diagram $E_n^\wedge \to L_A^{st} E \to E_n^\wedge$. Since $L_A^{st} E$ is L_A^{st}-local, the same is true for the retract $E_n^\wedge$, which is what we wanted to show. $\qquad\square$

Theorem 6.51. *For this theorem, assume that* $\mathrm{cd}_2(k) < \infty$.

(1) *Let* $X \in \mathcal{S}\mathrm{pc}(k)_*$ *be nilpotent. Then* $L_n X \in \mathcal{S}\mathrm{pc}(k)_*$ *(i.e.,* $L_n X$ *is again* $\mathbb{A}^1$*-invariant).*

(2) *Let* $X \in \mathcal{S}\mathrm{pc}(k)_*$ *be resolvable. There is a canonical map* $L_n X \to \tau_{\geq 1} \mathrm{Tot}\, T^\bullet X$, *which is an equivalence.*

(3) *For* $E \in \mathcal{SH}(k)_{\geq 1}$ *the canonical map* $L_n \Omega^\infty E \to \tau_{\geq 1} \Omega^\infty L_A^{st} E$ *is an equivalence.*

(4) *Let* $F \to E \to B$ *be a fiber sequence of pointed nilpotent Nisnevich sheaves. Then* $L_S F \simeq \tau_{\geq 1} \mathrm{fib}(L_S E \to L_S B)$.

(5) *Let* $X_\bullet$ *be a tower of pointed nilpotent Nisnevich sheaves with* $X = \lim_i X_i$. *Assume that the tower is locally highly connected subordinate to* $\mathcal{U}$, *e.g. it is highly connected. Then* $L_n X \simeq \lim_i L_n X_i$.

Proof. (1), (4) and (5) were already shown in Theorem 6.41.

For (3) we combine Lemma 6.37 and Lemma 6.50.

Hence, the only thing left is (2). Exactly as in the proof of Theorem 6.41 we reduce to the case that $X = \Omega^\infty \Sigma^k M$ for $M \in \Sigma^{1,1} \mathcal{SH}(k)^{\mathrm{eff}\heartsuit}$ such that n acts by zero on M.

Considering the fiber sequence $M \to M[1/2] \to C$ (and potentially further splitting up C into two objects in the heart), we may furthermore assume that either M is 2-periodic or 2-power-torsion. One now argues exactly as in Theorem 6.48 to show that in both cases the canonical map $\Omega^\infty \Sigma^k M \to F \Omega^\infty \Sigma^k M$ is an equivalence. $\qquad\square$

Appendix A. Coherence data from adjunctions

In this section, let $L \colon \mathcal{C} \rightleftarrows \mathcal{D} \colon R$ be an adjunction between ∞-categories. Alternatively, by work of Riehl–Verity [RV15] and Haugseng [Hau21, Theorem 1.1], this adjunction is classified by a functor $\mathcal{M} \colon \mathcal{ADJ} \to \mathrm{Cat}_\infty$ of $(\infty, 2)$-categories, where Cat_∞ is the $(\infty, 2)$-category of ∞-categories, and $\mathcal{ADJ}$ is the free living adjunction [SS86, Aud74], which is a $(2, 2)$-category. In particular, recall that $\mathcal{ADJ}$ has two objects 0 and 1, and there are morphisms $l \colon 0 \to 1$ and $r \colon 1 \to 0$, such that $\mathcal{M}(0) = \mathcal{C}$, $\mathcal{M}(1) = \mathcal{D}$, $\mathcal{M}(l) = L$ and $\mathcal{M}(r) = R$. Moreover, there are 2-morphisms $u \colon \mathrm{id}_0 \to rl$ and $c \colon lr \to \mathrm{id}_1$ that map under $\mathcal{M}$ to the unit and counit of the adjunction $L \dashv R$.

In this section, we explain how one can use this point of view to extract simplicial data out of this adjunction, such as, for example, the cobar resolution or the codegeneracy cube. As far as we are aware, some of the results presented in this appendix have not appeared in the literature in the setting of ∞-categories, even though they are well-known to experts.

In the following, $\mathrm{MAP}_\mathbb{E}(-, -)$ denotes the mapping ∞-category in an $(\infty, 2)$-category $\mathbb{E}$. This appendix is the only part of the article where we make (light) use of the theory of $(\infty, 2)$-categories.

Construction A.1 (Cobar resolution). We have $\mathrm{MAP}_{\mathcal{ADJ}}(0, 0) \simeq \Delta_+$ [Aud74, Corollary 2.8]. This lets us define an augmented cosimplicial object $(\mathrm{CB})_+^\bullet \colon \Delta_+ \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ as the composition

$$\Delta_+ \simeq \mathrm{MAP}_{\mathcal{ADJ}}(0, 0) \xrightarrow{\mathcal{M}} \mathrm{MAP}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathcal{C}) = \mathrm{Fun}(\mathcal{C}, \mathcal{C}).$$

Remark A.2. By [Aud74, Corollary 2.8], the object $[n] \in \Delta_+$ corresponds to the morphism $(rl)^n \in \mathrm{MAP}_{\mathcal{ADJ}}(0, 0)$. Therefore, $(\mathrm{CB})_+^n = \mathcal{M}((rl)^n) = (RL)^n$. Similarly, one sees that the transition morphisms in the cosimplicial diagram are given by (compositions of) the co/unit of the adjunction.

Recall the categories $\Delta_{+\infty}$ and $\Delta_{-\infty}$ from [Lur09, Lemma 6.1.3.16] and [Lur17, Definition 4.7.2.1]. An augmented cosimplicial object $X^\bullet \colon \Delta_+ \to \mathcal{C}$ is called *split* if it extends along the inclusion $\Delta_+ \subset \Delta_{+\infty}$ [Lur17, Definition 4.7.2.2], or equivalently along the inclusion $\Delta_+ \subset \Delta_{-\infty}$ [Lur17, Remark 4.7.2.3].

Remark A.3. Split objects have the useful property that they are automatically limit diagrams [Lur09, dual of Lemma 6.1.3.16], and the limit is preserved by any functor (since any functor preserves the extension and thus split objects).

Proposition A.4. *Both* $L(\mathrm{CB})_+^\bullet$ *and* $(\mathrm{CB})_+^\bullet R$ *are split cosimplicial objects. In particular, they are limit diagrams which are preserved by any functor.*
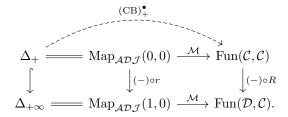
Proof. We start with the statement about $L(\mathrm{CB})_+^\bullet$. Consider the following diagram:

$$
\begin{array}{ccc}
& \overset{(\mathrm{CB})^\bullet_+}{\dashrightarrow} & \\
\Delta_+ === \mathrm{Map}_{\mathcal{ADJ}}(0,0) & \xrightarrow{\mathcal{M}} & \mathrm{Fun}(\mathcal{C},\mathcal{C}) \\
\downarrow \qquad\quad \downarrow{\scriptstyle l\circ(-)} & & \downarrow{\scriptstyle L\circ(-)} \\
\Delta_{-\infty} === \mathrm{Map}_{\mathcal{ADJ}}(0,1) & \xrightarrow{\mathcal{M}} & \mathrm{Fun}(\mathcal{C},\mathcal{D}).
\end{array}
$$

The squares commute essentially by [Aud74, Corollary 2.8] and since $\mathcal{M}$ is a 2-functor. The composition on the top and then the right is $L(\mathrm{CB})^\bullet_+$. Since the diagram commutes, we see that this has an extension to $\Delta_{-\infty}$, thus is split. For the other statement, consider instead the following diagram:

$$
\begin{array}{ccc}
& \overset{(\mathrm{CB})^\bullet_+}{\dashrightarrow} & \\
\Delta_+ === \mathrm{Map}_{\mathcal{ADJ}}(0,0) & \xrightarrow{\mathcal{M}} & \mathrm{Fun}(\mathcal{C},\mathcal{C}) \\
\downarrow \qquad\quad \downarrow{\scriptstyle (-)\circ r} & & \downarrow{\scriptstyle (-)\circ R} \\
\Delta_{+\infty} === \mathrm{Map}_{\mathcal{ADJ}}(1,0) & \xrightarrow{\mathcal{M}} & \mathrm{Fun}(\mathcal{D},\mathcal{C}).
\end{array}
$$

The squares commute essentially by (the dual of) [Aud74, Corollary 2.8] and since $\mathcal{M}$ is a 2-functor. The composition on the top and then the right is $(\mathrm{CB})^\bullet_+ R$. Since the diagram commutes, we see that this has an extension to $\Delta_{+\infty}$, thus is split. $\qquad\square$

Construction A.5. Write $\Delta^\pm \subset \Delta$ for the wide subcategory spanned by those maps that preserve both the minimal and maximal element. Moreover, write $\Delta^{\mathrm{inj}}_+ \subset \Delta_+$ and $\Delta^{\mathrm{surj}} \subset \Delta$ for the wide subcategories of injective, respectively surjective maps. In particular, $\Delta^{\mathrm{surj}} \subset \Delta^\pm$. Observe that there is a canonical equivalence $\Phi\colon \Delta^\pm \xrightarrow{\simeq} (\Delta_+)^{\mathrm{op}}$ [Str80, §3.11]. This equivalence restricts to an equivalence $\Phi\colon \Delta^{\mathrm{surj}} \xrightarrow{\simeq} (\Delta^{\mathrm{inj}}_+)^{\mathrm{op}}$. For $n \geq 0$, slicing over $[n]$ induces an equivalence $\Phi_n\colon \Delta^{\mathrm{surj}}_{[n]/} \xrightarrow{\simeq} (\Delta^{\mathrm{inj}}_{+,/[n-1]})^{\mathrm{op}}$.

Definition A.6. For $n \geq 0$ and $\mathcal{E}$ an ∞-category, write $\mathrm{Cube}^n(\mathcal{E}) := \mathrm{Fun}(\Delta^{\mathrm{surj}}_{[n]/}, \mathcal{E})$ for the ∞-category of *cubes* in $\mathcal{E}$.

Definition A.7. Let $n \geq 0$. We define two cubes as the following compositions:

$$
(\mathrm{CB})^\bullet_{\mathrm{surj},n}\colon \Delta^{\mathrm{surj}}_{[n]/} \to \Delta^{\mathrm{surj}} \subset \Delta_+ = \mathrm{MAP}_{\mathcal{ADJ}}(0,0) \xrightarrow{\mathcal{M}} \mathrm{Fun}(\mathcal{C},\mathcal{C}),
$$

so that $(\mathrm{CB})^\bullet_{\mathrm{surj},n} \in \mathrm{Cube}^n(\mathrm{Fun}(\mathcal{C},\mathcal{C}))$, and

$$
(\mathrm{CB})^\bullet_{+,\mathrm{inj},n}\colon \Delta^{\mathrm{surj}}_{[n]/} \xrightarrow[\simeq]{\Phi_n} (\Delta^{\mathrm{inj}}_{+,/[n-1]})^{\mathrm{op}} \to (\Delta^{\mathrm{inj}}_+)^{\mathrm{op}} \subset (\Delta_+)^{\mathrm{op}} = \mathrm{MAP}_{\mathcal{ADJ}}(1,1) \xrightarrow{\mathcal{M}} \mathrm{Fun}(\mathcal{D},\mathcal{D}),
$$

so that $(\mathrm{CB})^\bullet_{+,\mathrm{inj},n} \in \mathrm{Cube}^n(\mathrm{Fun}(\mathcal{D},\mathcal{D}))$. We call $(\mathrm{CB})^\bullet_{\mathrm{surj},n}$ the n-th *codegeneracy cube* and $(\mathrm{CB})^\bullet_{+,\mathrm{inj},n}$ the n-th *face cube* of the adjunction $L \dashv R$.

Definition A.8. Let $\mathcal{E}$ be an ∞-category and $n \geq 0$. We define a functor $\iota_{\mathrm{top}}\colon \Delta^{\mathrm{surj}}_{[n]/} \to \Delta^{\mathrm{surj}}_{[n+1]/}$ via

$$
(f\colon [n] \twoheadrightarrow [k]) \mapsto ([n+1] = [n] \star [0] \xrightarrow{f \star \mathrm{id}} [k] \star [0] = [k+1]),
$$

where $\star$ is the join operation. Precomposition with ι_{top} gives a functor

$$
(-)^{\mathrm{top}}\colon \mathrm{Cube}^{n+1}(\mathcal{E}) \to \mathrm{Cube}^n(\mathcal{E}),
$$

the *top face* of a cube.

   Similarly, we define $\iota_{\mathrm{bot}}\colon \Delta^{\mathrm{surj}}_{[n]/} \to \Delta^{\mathrm{surj}}_{[n+1]/}$ via precomposition with the surjective function $[n+1] \to [n]$ that is the identity on $0 \leq k \leq n$ and sends $n+1$ to n. Precomposition with ι_{bot} gives a functor

$$
(-)^{\mathrm{bot}}\colon \mathrm{Cube}^{n+1}(\mathcal{E}) \to \mathrm{Cube}^n(\mathcal{E}),
$$

the *bottom face* of a cube.

   There is a natural transformation $\iota_{\mathrm{top}} \to \iota_{\mathrm{bot}}$ given on $f\colon [n] \twoheadrightarrow [k]$ by the square

$$
\begin{array}{ccc}
[n+1] & \xrightarrow{\iota_{top}(f)} & [k+1] \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle i\mapsto i, k+1\mapsto k} \\
[n+1] & \xrightarrow{\iota_{bot}(f)} & [k],
\end{array}
$$

and hence an induced natural transformation

$$\psi \colon (-)^{\mathrm{top}} \to (-)^{\mathrm{bot}}.$$

This lets us inductively define the total fiber of a cube:

Definition A.9. Let $\mathcal{E}$ be an ∞-category with finite limits, and $n \geq 0$. We inductively define a functor $\mathrm{ToFib} \colon \mathrm{Cube}^n(\mathcal{E}) \to \mathcal{E}$ as follows: If $n = 0$, we use the equivalence $\mathrm{ToFib} \colon \mathrm{Cube}^n(\mathcal{E}) \simeq \mathcal{E}$ given by evaluating at $[0]$. If $n \geq 1$, then we define

$$\mathrm{ToFib}(-) \coloneqq \mathrm{fib}(\mathrm{ToFib}((-)^{\mathrm{top}}) \xrightarrow{\psi} \mathrm{ToFib}((-)^{\mathrm{bot}})).$$

We write $C^n \coloneqq \mathrm{ToFib}((\mathrm{CB})^\bullet_{\mathrm{surj},n}) \colon \mathcal{C} \to \mathcal{C}$ and $S^n \coloneqq \mathrm{ToFib}((\mathrm{CB})^\bullet_{+,\mathrm{inj},n}) \colon \mathcal{D} \to \mathcal{D}$ for the total fibers of the codegeneracy and face cubes.

Proposition A.10. *There is a canonical equivalence of cubes* $(\mathrm{CB})^\bullet_{\mathrm{surj},n} \simeq R(\mathrm{CB})^\bullet_{+,\mathrm{inj},n}L$, *and hence also of their total fibers* $C^n \simeq RS^nL$.

Proof. We have the following diagram:

$$
\begin{array}{ccccccccc}
 & & & & (\mathrm{CB})^\bullet_{\mathrm{surj},n} & & & & \\
\Delta^{\mathrm{surj}}_{[n]/} & \dashrightarrow & \Delta^{\mathrm{surj}} & \hookrightarrow & \Delta^{\pm} & \hookrightarrow & \Delta_+ & = & \mathrm{Map}_{\mathcal{ADJ}}(0,0) \xrightarrow{\mathcal{M}} \mathrm{Fun}(\mathcal{C},\mathcal{C}) \\
\Phi_n \downarrow \simeq & & \Phi \downarrow \simeq & & \Phi \downarrow \simeq & & & & r\circ(-)\circ l \uparrow \qquad R\circ(-)\circ L \uparrow \\
(\Delta^{\mathrm{inj}}_{+,/[n-1]})^{\mathrm{op}} & \longrightarrow & (\Delta^{\mathrm{inj}}_+)^{\mathrm{op}} & \hookrightarrow & (\Delta_+)^{\mathrm{op}} & = & & = & \mathrm{Map}_{\mathcal{ADJ}}(1,1) \xrightarrow{\mathcal{M}} \mathrm{Fun}(\mathcal{D},\mathcal{D}). \\
 & & & & (\mathrm{CB})^\bullet_{+,\mathrm{inj},n}\circ\Phi_n^{-1} & & & &
\end{array}
$$

Here, the left two squares commute by the definitions of Φ_n and Φ. That the rectangle commutes can be seen by comparing the definition of Φ with the explicit description of the morphism $r \circ (-) \circ l$, cf. [Aud74, Corollary 2.8]. The right square commutes since $\mathcal{M}$ is a 2-functor. Thus, the composition on the top agrees with the composition on the left, bottom and right. This is exactly what we want to prove. $\qquad\square$

Proposition A.11. *There is a fiber sequence*

$$S^{n+1} \to S^n LR \to S^n,$$

where the second map is induced by the counit of the adjunction $L \dashv R$.

Proof. By definition there is a fiber sequence $S^{n+1} \to \mathrm{ToFib}(\mathrm{CB})^{\bullet,\mathrm{top}}_{+,\mathrm{inj},n} \to \mathrm{ToFib}(\mathrm{CB})^{\bullet,\mathrm{bot}}_{+,\mathrm{inj},n}$. Therefore, it suffices to identify $(\mathrm{CB})^{\bullet,\mathrm{top}}_{+,\mathrm{inj},n} \simeq (\mathrm{CB})^\bullet_{+,\mathrm{inj},n-1}LR$ and $(\mathrm{CB})^{\bullet,\mathrm{bot}}_{+,\mathrm{inj},n} \simeq (\mathrm{CB})^\bullet_{+,\mathrm{inj},n-1}$. We start with the first claim, for which we have to show that the following diagram commutes:

$$
\begin{array}{ccccccccc}
\Delta^{\mathrm{surj}}_{[n-1]/} & \xrightarrow{\Phi_{n-1}} & \Delta^{\mathrm{inj,op}}_{+,/[n-2]} & \longrightarrow & \Delta^{\mathrm{op}}_+ & = & \mathrm{MAP}_{\mathcal{ADJ}}(1,1) & \xrightarrow{\mathcal{M}} & \mathrm{Fun}(\mathcal{D},\mathcal{D}) \\
\downarrow \iota_{\mathrm{top}} & & & & & & \downarrow lr\circ- & & \downarrow LR\circ- \\
\Delta^{\mathrm{surj}}_{[n]/} & \xrightarrow{\Phi_n} & \Delta^{\mathrm{inj,op}}_{+,/[n-1]} & \longrightarrow & \Delta^{\mathrm{op}}_+ & = & \mathrm{MAP}_{\mathcal{ADJ}}(1,1) & \xrightarrow{\mathcal{M}} & \mathrm{Fun}(\mathcal{D},\mathcal{D}).
\end{array}
$$

The left rectangle commutes by the explicit description of the morphism $lr\circ-$, cf. [Aud74, Corollary 2.8]. The right square commutes since $\mathcal{M}$ is a 2-functor. The second claim follows from the commutativity of the diagram

$$
\begin{array}{ccc}
\Delta^{\mathrm{surj}}_{[n-1]/} & \xrightarrow{\Phi_{n-1}} & \Delta^{\mathrm{inj,op}}_{+,/[n-2]} \\
\downarrow \iota_{\mathrm{bot}} & & \searrow \\
\Delta^{\mathrm{surj}}_{[n]/} & \xrightarrow{\Phi_n} \ \Delta^{\mathrm{inj,op}}_{+,/[n-1]} \longrightarrow & \Delta^{\mathrm{inj,op}}_+,
\end{array}
$$

which is immediate from the definition. $\qquad\square$

APPENDIX B. BOUSFIELD LOCALIZATIONS IN STABLE CATEGORIES AT MOORE OBJECTS

In this section, we prove the following proposition:

Proposition B.1. *Let $\mathcal{D}$ be a stable presentably symmetric monoidal ∞-category with unit $\mathbb{1}$, and $\mathcal{D}_{\geq 0} \subset \mathcal{D}$ the connective part of a t-structure that is left-complete (cf. the discussion before [Lur17, Remark 1.2.1.18]) and compatible with the symmetric monoidal structure and filtered colimits. Suppose that $A \in \mathrm{CAlg}(\mathcal{D}_{\geq 0})$. Write L for the Bousfield localization at those morphisms f such that $f \otimes A$ is an equivalence. Suppose moreover that there are maps $x_i \colon L_i \to \mathbb{1}$ such that $L_i \in \mathcal{D}_{\geq 0}$ is dualizable, with dual $DL_i \in \mathcal{D}_{\geq 0}$. Let $S \subset \mathbb{Z} \setminus 0$ be a set of nonzero integers. If $\pi_0(A) \simeq S^{-1}\pi_0(\mathbb{1}/\!\!/(x_1, \ldots, x_n))$, then $LE \simeq (S^{-1}E)^\wedge_{x_1,\ldots,x_n}$ for all $E \in \mathcal{D}_{\geq 0}$.*

Proof. As the t-structure on $\mathcal{D}$ is compatible with filtered colimits, it restricts to a t-structure on $\mathcal{D}[S^{-1}]$ such that the localization functor $S^{-1}(-)$ is t-exact. It is moreover symmetric monoidal, so that $S^{-1}L_i \in \mathcal{D}[S^{-1}]_{\geq 0}$ is dualizable, with dual $S^{-1}DL_i \in \mathcal{D}[S^{-1}]_{\geq 0}$. Note that $A \in \mathcal{D}[S^{-1}]$ since S is invertible on $\pi_0(A)$ by assumption. Let $E \in \mathcal{D}_{\geq 0}$. Since $\pi_0(A) = S^{-1}\pi_0(\mathbb{1}/\!\!/(x_1, \ldots, x_n)) = \pi_0((S^{-1}\mathbb{1})/\!\!/(x_1, \ldots, x_n))$, we conclude from [BØ22, Theorems 2.1 and 2.2] that $(S^{-1}E)^\wedge_{x_1,\ldots,x_n} \simeq \lim(A \otimes S^{-1}E \rightrightarrows A \otimes A \otimes S^{-1}E \cdots)$ is given by the A-nilpotent completion. Consider the canonical map $E \to S^{-1}E \to (S^{-1}E)^\wedge_{x_1,\ldots,x_n}$. Since the nilpotent completion is L-local (as a limit of A-modules), the right-hand side is L-local. Thus, as the map is an $(- \otimes A)$-equivalence, we get an equivalence $LE \simeq (S^{-1}E)^\wedge_{x_1,\ldots,x_n}$. $\qquad\square$

APPENDIX C. UNSTABLE n-COMPLETION

In this appendix we discuss unstable completion at a (possibly infinite) set of primes.

Lemma C.1. *Let $\mathcal{D}$ be a stable presentable ∞-category, and $n \in \mathbb{Z} \setminus 0$. Then $E_n^\wedge \simeq \bigoplus_{\ell \mid n} E_\ell^\wedge$, where the product is over all primes ℓ dividing n. In particular, $E_\ell^\wedge$ is n-complete.*

Proof. First, f is an n-equivalence if and only if f is an ℓ-equivalence for all $\ell \mid n$. Indeed, consider the devissage fiber sequence $f/\!\!/\ell \to f/\!\!/n \to f/\!\!/\frac{n}{\ell}$, and conclude by induction. In particular, $(-)_n^\wedge$ inverts all ℓ-equivalences, therefore there is a canonical functor $(-)_n^\wedge \to \bigoplus_{\ell \mid n}(-)_\ell^\wedge$.

We show that this map is an n-equivalence. Indeed, since the map $E \to E_n^\wedge$ is an n-equivalence, it suffices to show that $E \to \bigoplus_{\ell \mid n} E_\ell^\wedge$ is an n-equivalence, or equivalently an p-equivalence for all $p \mid n$. This now follows since $(E_p^\wedge)_p^\wedge \simeq E_p^\wedge$, and $(E_\ell^\wedge)_p^\wedge = 0$ for all $\ell \neq p$ since p is invertible on $E_\ell^\wedge$.

We finish the proof by showing that any object of the form $\bigoplus_{\ell \mid n} E_\ell^\wedge$ is n-complete. Indeed, it suffices to show that $E_\ell^\wedge$ is n-complete for all $\ell \mid n$, which is clear since it is local for all ℓ-equivalences, and every n-equivalence is in particular an ℓ-equivalence.

For the last claim, note that n-complete objects are stable under retracts. $\qquad\square$

Let $\mathcal{P}$ be a (not necessarily finite) set of primes and $\mathcal{X}$ be an ∞-topos. Suppose that $\mathcal{X}$ admits a locally finite-dimensional cover, as in [Mat24a, Definition 5.1].

Definition C.2. Write $L_\mathcal{P} \colon \mathcal{X} \to \mathcal{X}$ for the Bousfield-localization at $\mathcal{P}$-equivalences, i.e., at morphisms $f \colon X \to Y$ such that $\Sigma^\infty_+ f/\!\!/\ell$ is an equivalence for all $\ell \in \mathcal{P}$. We call this functor the *unstable $\mathcal{P}$-completion functor*.

If $n \in \mathbb{Z} \setminus 0$ is a nonzero integer, we write $\mathcal{P}_n$ for the set of prime divisors of n, and $L_n \coloneqq L_{\mathcal{P}_n}$ for the *unstable n-completion functor*.

In particular, we have the following:

Example C.3. If $n = \ell^k$ is a prime power (for $k > 0$), then $L_n = L_\ell$ is just unstable ℓ-completion. If $n = 1$, then $L_n = *$, as then any morphism is an n-equivalence. Similarly, $L_\emptyset = *$, since $\mathcal{P}_1 = \emptyset$.

The following is straightforward:

Lemma C.4. *Let $n \in \mathbb{Z} \setminus 0$ and $f \colon X \to Y \in \mathcal{X}$. Then f is an n-equivalence (i.e. $\Sigma^\infty_+ f/\!\!/n$ is an equivalence) if and only if f is an ℓ-equivalence for all $\ell \mid n$ (i.e. $L_{\mathcal{P}_n} f$ is an equivalence).*

Proof. This follows since it is true stably, cf. Lemma C.1. $\qquad\square$

Our main goal is to show that if X is nilpotent, then $L_\mathcal{P} X$ splits into the product of $L_\ell X$. First, we get a natural transformation between the two functors in question:

Lemma C.5. *There is a canonical natural transformation $\varphi \colon L_\mathcal{P} \to \prod_{\ell \in \mathcal{P}} L_\ell$.*

If $X \in \mathcal{X}_*$ is connected, then so is $L_\mathcal{P} X$. In particular, we obtain by adjunction a natural morphism $\widetilde{\varphi}_X \colon L_\mathcal{P} X \to \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X$.

Proof. For the first statement, it suffices to give natural transformations $L_{\mathcal{P}} \to L_\ell$ if $\ell \in \mathcal{P}$. Since by definition any $\mathcal{P}$-equivalence is also an ℓ-equivalence, we see that L_ℓ inverts all $\mathcal{P}$-equivalences, and hence the canonical map $\mathrm{id} \to L_\ell$ factors over $L_{\mathcal{P}}$.

Now to prove the second statement, let $X \in \mathcal{X}_*$ be connected. We must show the same is true for $L_{\mathcal{P}}X$. If $\mathcal{P} = \emptyset$ then $L_{\mathcal{P}}X = *$, which is certainly connected. Otherwise, there exists $\ell \in \mathcal{P}$. By definition, $X \to L_{\mathcal{P}}X$ is an ℓ-equivalence, and thus an equivalence on π_0 by [Mat24b, Lemma 3.12]. Hence, again $L_{\mathcal{P}}X$ is connected. $\qquad\square$

We can now state the main result:

Theorem C.6. *Let $X \in \mathcal{X}_*$ be nilpotent. Then $\widetilde{\varphi}_X$ is an equivalence.*
If $\mathcal{P}$ is finite, then $\prod_{\ell \in \mathcal{P}} L_\ell X$ is connected and in particular φ_X is an equivalence.
As a special case, if $n \in \mathbb{Z} \setminus 0$, then $L_n X \simeq \prod_{\ell \mid n} L_\ell X$.

For the proof we need some preparations. The following definition will be useful.

Definition C.7. Let $X \in \mathcal{X}_*$ be nilpotent. A *Postnikov refinement* $(X_n, E_n)_n$ is a tower of nilpotent sheaves $(X_n)_n$ under X, that fit into fiber sequences $X_{n+1} \to X_n \to \Omega_*^\infty E_n$ for some $E_n \in \Sigma^{k_i} \mathrm{Sp}(\mathcal{X})^\heartsuit$ with $k_i \geq 2$, such that moreover $X_0 = *$, $X \simeq \lim_n X_n$, and $k_i \to \infty$ as $i \to \infty$.

Remark C.8. If $X \in \mathcal{X}_*$ is nilpotent, then there always exists a Postnikov refinement, cf. [Mat24b, Lemma A.15]. Moreover, if $(X_n, E_n)_n$ is a Postnikov refinement of X, then $(X_n)_n$ is a highly connected tower.

Lemma C.9. *Let $X \in \mathcal{X}_*$ be nilpotent, and let $(X_n, E_n)_n$ be a Postnikov refinement of X. Then the following holds:*
(1) $\tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell \Omega^\infty E_n \simeq \tau_{\geq 1} \Omega^\infty \prod_{\ell \in \mathcal{P}} (E_n)_\ell^\wedge$.
(2) $\tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X_{n+1} \simeq \tau_{\geq 1} \mathrm{fib}(\tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X_n \to \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell \Omega^\infty E_n)$ for every n.

Proof. For (1) we have

$$\tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell \Omega^\infty E_n \simeq \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} \tau_{\geq 1} \Omega^\infty (E_n)_\ell^\wedge \simeq \tau_{\geq 1} \Omega^\infty \prod_{\ell \in \mathcal{P}} (E_n)_\ell^\wedge,$$

using [Mat24b, Lemma 3.17] and [Mat24a, Lemma 4.2].

For (2) we get

$$\tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X_{n+1} \simeq \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} \tau_{\geq 1} \mathrm{fib}(L_\ell X_n \to L_\ell \Omega^\infty E_n) \simeq \tau_{\geq 1} \mathrm{fib}(\tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X_n \to \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell \Omega^\infty E_n),$$

using [Mat24b, Proposition 3.20] and [Mat24a, Lemma 4.2]. $\qquad\square$

Lemma C.10. *Let $X \in \mathcal{X}_*$ be nilpotent, and let $(X_n, E_n)_n$ be a Postnikov refinement of X. Then for every $n \geq 0$ the sheaf $\tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X_n$ is nilpotent.*

Proof. We show the claim by induction on n, the case $n = 0$ holds trivially. For any $n \geq 0$ we get $\tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X_{n+1} \simeq \tau_{\geq 1} \mathrm{fib}(\tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X_n \to \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell \Omega^\infty E_n)$ from Lemma C.9 (2). The first term on the right is nilpotent by induction, whereas the second is by combining Lemma C.9 (1) with [Mat24b, Lemma A.11]. Hence, we conclude using [Mat24b, Lemma A.12]. $\qquad\square$

Lemma C.11. *Let $\mathcal{U}$ be an ∞-topos generated under colimits by a set $W \subset \mathcal{U}$. Let $X \in \mathcal{U}_*$ be an object and $n \in \mathbb{N}$ such that $X(w)$ is an n-connective space for every $w \in W$. Then X is n-connective in $\mathcal{U}_*$.*

Proof. Consider the adjunction $L \colon \mathcal{P}(W) \rightleftarrows \mathcal{U} \colon i$, where L is the left Kan extension of $W \hookrightarrow \mathcal{U}$, and i is the restricted Yoneda. Since W generates $\mathcal{U}$ under colimits, it follows that L is essentially surjective, and i is fully faithful, so that $Li(X) \simeq X$. Moreover, note that since i preserves the terminal object (as a right adjoint), we also get $L(*) = Li(*) = *$. By assumption on X, $i(X)$ is an n-connective object of $\mathcal{P}(W)_*$ (note that in a presheaf topos n-connective objects are exactly those presheaves such that they are n-connective on sections).

It now suffices to show that any functor $F \colon \mathcal{V} \to \mathcal{U}$ between ∞-topoi that preserves colimits and the final object also preserves n-connective objects. Indeed, for this note first that F preserves suspensions, i.e. $\Sigma^n FV \simeq F\Sigma^n V$, since suspensions are iterated pushouts of the form $* \leftarrow V \to *$. Thus, the claim follows since any n-connective object in $\mathcal{V}_*$ can be written as a colimit of objects of the form $\Sigma^n V$, cf. [Mat25, Proposition 2.38]. $\qquad\square$

Lemma C.12. *Let $\mathcal{U}$ be an ∞-topos locally of homotopy dimension $\leq N$ for some $N \in \mathbb{N}$. Let $k > N$ and $X_\ell \in \mathcal{U}_*$ be a k-connective object for every $\ell \in \mathcal{P}$. Then $\prod_{\ell \in \mathcal{P}} X_\ell$ is $(k - N)$-connective.*

Proof. It suffices to show that $(\prod_{\ell \in \mathcal{P}} X_\ell)(U)$ is $(k-N)$-connective for every $U \in \mathcal{U}$ of homotopy dimension $\leq N$ (by Lemma C.11, since they generate the topos under colimits). By [Lur09, Lemma 7.2.1.7] (applied to the slice topos $\mathcal{U}_{/U}$), we have that $X_\ell(U)$ is $(k-N)$-connective. This immediately implies the lemma since evaluation of sheaves commutes with limits, and arbitrary products of $(k-N)$-connective spaces are $(k-N)$-connective. $\square$

Remark C.13. In the last lemma, if $\mathcal{P}$ is finite, then of course $\prod_{\ell \in \mathcal{P}} X_\ell$ is even k-connective.

Lemma C.14. *Let* $X \in \mathcal{X}_*$ *be nilpotent, and let* $(X_n, E_n)_n$ *be a Postnikov refinement of* X. *The tower* $(\tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X_n)_n$ *is locally highly connected (subordinate to any highly connected cover of* $\mathcal{X}$*), with limit* $\tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X$.

Proof. Choose a highly connected cover $\mathcal{U} = \{p_i^* \colon \mathcal{X} \to \mathcal{U}_i\}$. For the first statement, it suffices to show that $(\prod_{\ell \in \mathcal{P}} L_\ell X_n)_n$ is locally highly connected. Since the p_i^* commute with both p-completion and limits by [Mat24a, Lemma 6.10 and Lemma 5.2], we may assume that $\mathcal{X}$ itself is of homotopy dimension $\leq N$ for some $N \in \mathbb{N}$, and that $(X_n)_n$ is a highly connected tower, and our goal is to show that also $(\prod_{\ell \in \mathcal{P}} L_\ell X_n)_n$ is highly connected. So let $k \in \mathbb{N}$. Since $(X_n)_n$ is highly connected, there exists $L \geq 0$ such that for all $m \geq L$ the fiber $\mathrm{fib}(X_m \to X_L)$ is $(k+2N+3)$-connective. We have to find $M \geq 0$ such that for all $m \geq M$ the object $\prod_{\ell \in \mathcal{P}} \mathrm{fib}(L_\ell X_m \to L_\ell X_M)$ is $(k+1)$-connective (since we can commute the product with the fiber). Setting $M = L$, it follows from [Mat25, Lemma 2.11] that $\mathrm{fib}(L_\ell X_m \to L_\ell X_M)$ is $(k+1+N)$-connective for every $\ell \in \mathcal{P}$, and hence from Lemma C.12 that $\prod_{\ell \in \mathcal{P}} \mathrm{fib}(L_\ell X_m \to L_\ell X_M)$ is $(k+1)$-connective, proving the claim.

It is left to compute the limit of the tower, where we can again go to the highly connected cover. We have $\lim_n L_\ell X_n \simeq L_\ell X$ by [Mat25, Proposition 2.13], since $(X_n)_n$ is highly connected. Hence, we get $\tau_{\geq 1} \lim_n \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X_n \simeq \tau_{\geq 1} \lim_n \prod_{\ell \in \mathcal{P}} L_\ell X_n \simeq \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X$, using [Mat24a, Lemma 4.2]. To conclude, we show that $\lim_n \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X_n$ is already connected. This holds since $(\tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X_n)_n$ is highly connected by the first part of the proof, and thus $\pi_0(\lim_n \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X_n) = *$ by [Mat25, Lemma 2.4]. $\square$

Lemma C.15. *Let* $X \in \mathcal{X}_*$ *be nilpotent. Then* $\widetilde{\varphi}_X \colon L_{\mathcal{P}} X \to \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X$ *is a* $\mathcal{P}$*-equivalence.*

Proof. We have to see that $\widetilde{\varphi}_X$ is an p-equivalence for every $p \in \mathcal{P}$. By definition, we know that $X \to L_{\mathcal{P}} X$ is a p-equivalence, hence, by 2-out-of-3 it suffices to show that $X \to \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X$ is a p-equivalence. Note that $\tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X \simeq L_p X \times \tau_{\geq 1} \prod_{p \neq \ell \in \mathcal{P}} L_\ell X$. Since L_p preserves finite products [Mat24b, Lemma 3.15] and since $L_p L_p \simeq L_p$, it suffices to show that $L_p \tau_{\geq 1} \prod_{p \neq \ell \in \mathcal{P}} L_\ell X = *$.

Choose a Postnikov refinement $(X_n, E_n)_n$ of X. First, for every n we have

$$\tau_{\geq 1} \prod_{p \neq \ell \in \mathcal{P}} L_\ell \Omega^\infty E_n \simeq \Omega^\infty \tau_{\geq 1} \prod_{p \neq \ell \in \mathcal{P}} (E_n)_\ell^\wedge,$$

using Lemma C.9 (1). In order to see that the p-completion of this vanishes, using [Mat24b, Lemma 3.16] it is enough to show that $(\tau_{\geq 1} \prod_{p \neq \ell \in \mathcal{P}} (E_n)_\ell^\wedge)/\!\!/p = 0$. This holds since p is invertible on $(E_n)_\ell^\wedge$ for all $\ell \neq p$.

We now show inductively that $L_p \tau_{\geq 1} \prod_{p \neq \ell \in \mathcal{P}} L_\ell X_n = *$, the case $n = 0$ holds trivially. This follows immediately from Lemma C.9 (2), the inductive hypothesis and the case for E_n discussed above.

We finish the proof by showing that $L_p \tau_{\geq 1} \prod_{p \neq \ell \in \mathcal{P}} L_\ell X = *$. The tower $(\tau_{\geq 1} \prod_{p \neq \ell \in \mathcal{P}} L_\ell X_n)_n$ is locally highly connected, with limit $\tau_{\geq 1} \prod_{p \neq \ell \in \mathcal{P}} L_\ell X$, cf. Lemma C.14. Moreover, for every $n \geq 0$ the sheaf $\tau_{\geq 1} \prod_{p \neq \ell \in \mathcal{P}} L_\ell X_n$ is nilpotent by Lemma C.10. Hence, using [Mat25, Proposition 2.13] and the case for X_n discussed above, we get

$$L_p \tau_{\geq 1} \prod_{p \neq \ell \in \mathcal{P}} L_\ell X \simeq \lim_n L_p \tau_{\geq 1} \prod_{p \neq \ell \in \mathcal{P}} L_\ell X_n = *. \qquad \square$$

We are now ready to prove the main theorem of this section.

Proof of Theorem C.6. In Lemma C.15 we have seen that $\widetilde{\varphi}_X \colon L_{\mathcal{P}} X \to \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_\ell X$ is a $\mathcal{P}$-equivalence. Since the left-hand side is clearly $\mathcal{P}$-complete, it suffices to show that the right-hand side is $\mathcal{P}$-complete. Since $\mathcal{P}$-equivalences are ℓ-equivalences, limits of ℓ-complete objects (for varying ℓ) are $\mathcal{P}$-complete. Writing $\tau_{\geq 1} = \mathrm{fib}(\mathrm{id} \to \pi_0)$, this reduces to showing that $L_\ell X$ is ℓ-complete (which is obvious) and that discrete sheaves are ℓ-complete (which holds by [Mat24b, Lemma 3.13]).

For the second statement, i.e. the case where $\mathcal{P}$ is finite, we just note that $\prod_{\ell \in \mathcal{P}} L_\ell X$ is connected, since each of the finitely many factors is (see [Mat24b, Lemma 3.12] for a proof that $L_\ell X$ is connected). $\square$

Using the main theorem, we can now extend the results about unstable p-completion from [Mat24b, Mat24a, Mat25] to the unstable $\mathcal{P}$-completion functor. For $n \in \mathbb{Z} \setminus 0$, specializing to $\mathcal{P} = \mathcal{P}_n$ we also obtain analogous results about the unstable n-completion functor.

Lemma C.16 (Infinite loop spaces)**.** *Let* $E \in \mathrm{Sp}(\mathcal{X})_{\geq 1}$*. Then* $L_{\mathcal{P}} \Omega^{\infty} E \simeq \tau_{\geq 1} \Omega^{\infty} \prod_{\ell \in \mathcal{P}} E_{\ell}^{\wedge}$*. As a special case, we have* $L_n \Omega^{\infty} E \simeq \tau_{\geq 1} \Omega^{\infty} E_n^{\wedge}$*.*

Proof. We calculate

$$L_{\mathcal{P}} \Omega^{\infty} E \simeq \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_{\ell} \Omega^{\infty} E \simeq \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} \tau_{\geq 1} \Omega^{\infty} E_{\ell}^{\wedge} \simeq \tau_{\geq 1} \Omega^{\infty} \prod_{\ell \in \mathcal{P}} E_{\ell}^{\wedge},$$

where we used Theorem C.6 in the first equivalence, [Mat24b, Lemma 3.17] in the second equivalence, and that products commute with Ω^{∞} in the third equivalence.

The last claim holds since $\prod_{\ell \in \mathcal{P}_n} E_{\ell}^{\wedge} = \bigoplus_{\ell \in \mathcal{P}} E_{\ell}^{\wedge} \simeq E_n^{\wedge}$ by Lemma C.1. $\qquad\square$

Lemma C.17 (Fiber sequences)**.** *Let* $F \to X \to Y$ *be a fiber sequence of nilpotent sheaves in* $\mathcal{X}_*$*. Then* $L_{\mathcal{P}} F \simeq \tau_{\geq 1} \mathrm{fib}(L_{\mathcal{P}} X \to L_{\mathcal{P}} Y)$*. As a special case, we have* $L_n F \simeq \tau_{\geq 1} \mathrm{fib}(L_n X \to L_n Y)$*.*

Proof. We calculate

$$L_{\mathcal{P}} F \simeq \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_{\ell} F \simeq \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} \tau_{\geq 1} \mathrm{fib}(L_{\ell} X \to L_{\ell} Y)$$

$$\simeq \tau_{\geq 1} \mathrm{fib}(\tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_{\ell} X \to \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_{\ell} Y) \simeq \tau_{\geq 1} \mathrm{fib}(L_{\mathcal{P}} X \to L_{\mathcal{P}} Y),$$

where we used Theorem C.6 in the first and last equivalence, [Mat25, Proposition 2.8] in the second equivalence, and that products commute with fibers in the third equivalence. $\qquad\square$

Lemma C.18 (Highly connected towers)**.** *Let* $(X_i)_i$ *be a highly connected tower, where every* $X_i \in \mathcal{X}_*$ *is nilpotent. Then* $L_{\mathcal{P}} \lim_i X_i \simeq \lim_i L_{\mathcal{P}} X_i$*. As a special case, we have* $L_n \lim_i X_i \simeq \lim_i L_n X_i$*.*

Proof. First note that $\lim_i X_i$ is still nilpotent. This follows basically from Lemma 6.7, as then $\underline{\pi}_1(X) \simeq \underline{\pi}_1(X_N)$ for some $N \gg 0$, and hence is nilpotent by assumption, and similarly, the action of $\underline{\pi}_1(X)$ on $\underline{\pi}_n(X)$ is equivalent to the action of $\underline{\pi}_1(X_M)$ on $\underline{\pi}_n(X_M)$ for some $M \gg 0$, which is nilpotent by assumption. We calculate

$$L_{\mathcal{P}} \lim_i X_i \simeq \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_{\ell} \lim_i X_i \simeq \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} \lim_i L_{\ell} X_i \simeq \tau_{\geq 1} \lim_i \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_{\ell} X_i \simeq \tau_{\geq 1} \lim_i L_{\mathcal{P}} X_i,$$

where we used Theorem C.6 in the first and last equivalence, [Mat25, Proposition 2.13] in the second equivalence, and that products commute with limits in the third equivalence. The lemma follows since $\lim_i \tau_{\geq 1} \prod_{\ell \in \mathcal{P}} L_{\ell} X_i \simeq \lim_i L_{\mathcal{P}} X_i$ is already connected, the proof of which is similar to the end of the proof of Lemma C.14. $\qquad\square$

Appendix D. Slice convergence

In this section we explain a minor extension of Levine's convergence theorem, allowing us to treat imperfect extensions of perfect base fields. The argument mainly consists of observing that [BY20, Proposition 6.7] can be used as a slot-in replacement for a certain part of Levine's proof, and with this replacement, the original argument goes through in the more general setting.

Proposition D.1. *Let* k *be a perfect field,* $E \in \Sigma^{n,n} \mathcal{SH}(k)^{veff}$*, and* K/k *a finitely generated field extension. Then* $\underline{\pi}_0(E)_*(K)$ *for* $* \geq -n$ *is generated by transfers of elements in* $\underline{K}_*^{MW}(L) \cdot \underline{\pi}_0(E)_{-n}(L)$*, where* L/K *runs through finite field extensions.*

Proof. If $* = 0$ and $n = 1$ this is proved in [BY20, Proposition 6.7]. By shifting, this also proves $n = 1-*$ for any value of $*$. The general case follows from this and the projection formula. (Indeed we may as well assume that $n = 0$, then apply the previous case to $\Sigma^{r+1,r+1} E$ to learn that $\underline{\pi}_0(\Sigma^{r+1,r+1} E)_0 \simeq \underline{\pi}_0(E)_{r+1}$ is generated under transfers and $\underline{K}_1^{MW}$ by $\underline{\pi}_0(\Sigma^{r+1,r+1} E)_{-1} \simeq \underline{\pi}_0(E)_r$; now iterate.) $\qquad\square$

Corollary D.2. *There exists a function* $N \colon \mathbb{N}^2 \to \mathbb{N}$ *with the following property: For* k *a perfect field of characteristic* p*,* $m, n \geq 0$*,* $E \in \Sigma^{N(m,n),N(m,n)} \mathcal{SH}(k)^{veff}$ *p-torsion (i.e.* $E[1/p] = 0$*),* K/k *of transcendence degree at most* m*,* $i \leq n$ *and* $j \geq 0$*, we have*

$$\underline{\pi}_i(E)_j(K) = 0.$$

Proof. The proof is essentially the same as in [Lev13, Theorem 7.3] (also recalled in [BEØ24, Theorem 5.3]), with two key differences. Firstly, [Lev13, Theorem 5.3] is replaced by Proposition D.1. Secondly, in Levine's proof the Bloch–Kato conjecture is used to show that if L/k has transcendence degree $\leq m$, then $\underline{K}_i^{MW}(L)/p = 0$ for $i > m$. To conclude the proof, we supply a different argument to show that this still holds in our situation. Let us first show that $\underline{K}_i^M(L)/p = 0$ for all $i > m$. This can be seen as follows: There is an inclusion $\underline{K}_i^M(L)/p \hookrightarrow \Omega_L^i$ by the Bloch–Kato–Gabber theorem [BK86, Theorem 2.1]. But for $i > m$ we have $\Omega_L^i = \bigwedge_{l=1}^i \Omega_L^1 = 0$, since Ω_L^1 has dimension m [Mat80, Theorem 59]. We now note that modulo p the powers of the fundamental ideal of L vanish, i.e. $I^i(L)/p = 0$, for $i > m$. Indeed, if $p \neq 2$, then this holds since $W(L)$ has 2-power torsion (indeed, $W(L)$ is a module over $W(\mathbb{F}_p)$, which is of 2-power torsion, by the explicit calculation done in e.g. [Dég23, Example 2.1.10]), and hence p is invertible on $W(L)$ and therefore also on $I^i(L)$. On the other hand, if $p = 2$, we have $I^{i+1}(L)/I^i(L) \simeq \underline{K}_i^M(L)/2 = 0$ for $i > m$ by the Milnor Conjecture (see e.g. [Dég23, Theorem 2.2.3]) and the above vanishing result for Milnor K-theory. In particular, $I^{m+1}(L) = I^{m+2}(L) = \ldots$, which implies that all of these powers of the fundamental ideal vanish (since their intersection is 0 by the Arason–Pfister Hauptsatz [Lam05, Chapter 10, Corollary 3.2]). In view of the short exact sequence $0 \to I(L)^{i+1} \to \underline{K}_i^{MW}(L) \to \underline{K}_i^M(L) \to 0$ from e.g. [Dég23, Corollary 2.3.10 (1)], the snake lemma, and the above calculations, we thus get $\underline{K}_i^{MW}(L)/p \simeq \underline{K}_i^M(L)/p = 0$ for all $i > m$. $\square$

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Email address: tom.bachmann@zoho.com
Web-page: www.tom-bachmann.com

Email address: anton.engelmann@uni-mainz.de
Web-page: www.anton-engelmann.com

Email address: klaus.mattis@uni-mainz.de
Web-page: www.klaus-mattis.com